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STRENGTH MODELLING OF A RANDOMLY FISSURED MATERIAL

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Abstract: The paper deals with strong geomaterial samples weakened by a family of randomly distributed fissures (or cracks). Fissures are filled up with a weak Tresca-type material. The effect of strength reduction is analyzed employing a probabilistic modelling. Complete closed-form solutions are derived for both randomly isotropic and randomly anisotropic samples. The role of the number of fissures and their orientation (localisation) is discussed in numerical examples.

LIST OF SYMBOLS

a_1, a_2 – two predefined modal values of the bimodal pdf $f(\alpha)$ in equation (8), the most probable values of the angle α,
$A(\omega), A_i, X_i$ – random events, random variables,
A, B – two numerical parameters of the bimodal pdf $f(\alpha)$ of the orientation angle α, equation (8),
c_1, c_2 – cohesion of two materials considered ($c_1 = +\infty$ taken for simplification),
cpd – cumulative probability distribution, integrated pdf,
$f, f(\alpha)$ – structural parameter, given pdf of a fissure position versus orientation angle α,
$F, F(\alpha)$ – given cpd of a fissure position, integrated pdf, i.e. $F(\alpha) = \int f(a)da$,
F_n – cpd (to be found) for a fissured sample strength $R_c(\omega), \mathsf{F}_n(\rho) = P\{R_c/2c_2 < \rho\}$,
n – number of fissures (discontinuity lines) in a sample,
pdf – probability density function (non-negative, integrable to unity, usually continuous),
$\mathrm{P}\{A(\omega)\}$ – probability of a random event $A(\omega)$,
r – strength variable,
R_c, R_{c1}, R_{c2} – one-dimensional unconfined compression strength, $R_{ck} = 2c_k$, $k = 1, 2$,
α_i – angle, orientation of the i-th fissure in the structured sample,
α_ρ – angle, auxiliary numerical parameter, $0 \le \alpha_\rho = \arcsin(1/\rho)/2 \le \pi/4$,
ρ – auxiliary dimensionless variable, $\rho = r/(2c_2) \ge 1$,
ρ_c – auxiliary dimensionless variable, $\rho_c = R_c/(2c_2) \ge 1$,
σ_1 – one dimensional unconfined compression stress, $\sigma_1 > 0$,
ω – elementary random event.

1. OBJECTIVES

Let a geomaterial sample be composed of two Tresca materials:

1. The first one, called “rock blocks”, having cohesion c_1 and volume contribution f_1 [%],

2. The second one, called “soil filling”, having cohesion c_2 and volume contribution f_2 [%].

Both materials are homogeneous, isotropic and weightless. It is a well-established fact that the strength of the sample strongly depends on topology of the components.

Consider a 2D case, where the sample consists of random polygonal blocks of material No. 1 and a fissure filling material No. 2. Moreover, let $f_1 \sim 100\%$, $f_2 \sim 0\%$. Such a structure can happen if the material No. 1 is intersected by a family of thin lines of discontinuity presented in figure 1.

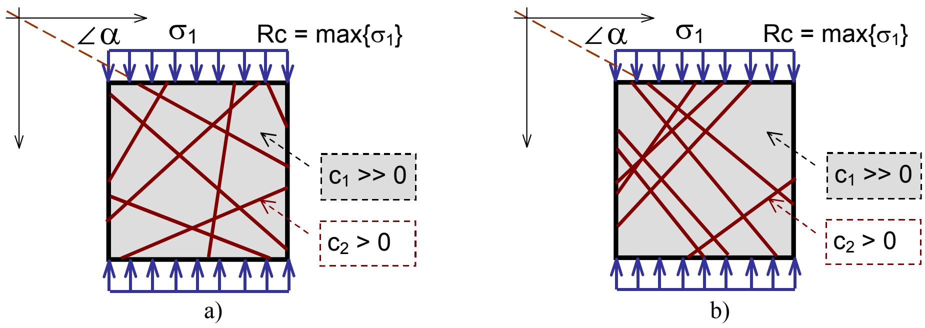

Fig. 1. Structure of a sample weakened by n random fissures ($c_2 < c_1$):
a) isotropic structure (in a mean sense), b) anisotropic structure.

Unconfined 1D compression strength equals $R_{ck} = 2c_k$ if both materials are considered separately. The analysis of $R_c = \max\{\sigma_1\}$ for a class of randomly structured samples is the objective of the paper. Fissure propagation (or closure) is not considered.

2. PROBABILISTIC ASSUMPTIONS

The orientation of each line l_i, $i = 1, 2, \ldots, n$, is defined by the angle α_i, which is a random variable being defined by its cpd (cumulative probability distribution) $F^{(i)}(a) = \mathrm{P}\{\omega\colon \alpha_i(\omega) < a\}$. All discontinuity lines (fissures, cracks) are assumed to be independent of each other, thus such are the variables α_i.

Assume that all cpds $F^{(i)}$ are the same, so $F^{(i)}(a) = F(a) = \mathrm{const}(i)$. If the random variables α_i have uniform distribution, i.e. constant pdf, then all orientations of the interface lines are equally probable. This leads – in a mean sense – to a randomly isotropic sample. If not, some directions are more or less privileged and the sample becomes randomly anisotropic.

As far as only one line of discontinuity is selected (figure 1), the strength R_c of the sample depends on c_1, c_2 and the interface angle α. The limit equilibrium for the angle $0 < \alpha < \pi/2$ yields

$$Rc = \min\left\{2c_1, \frac{2c_2}{\sin(2\alpha)}\right\} \tag{1a}$$

and for the whole spectrum of possible values $0 < \alpha < \pi$

$$Rc = \min\left\{2c_1, \frac{2c_2}{|\sin(2\alpha)|}\right\}. \tag{1b}$$

The absolute-value sign reflects mirror symmetries, since both cases α and $\pi - \alpha$ physically mean the same.

Define the following strength R_c of the sample in the presence of n lines of discontinuity

$$Rc = \min\left\{\min\left\{2c_1, \frac{2c_2}{|\sin(2\alpha_1)|}\right\}, \quad \ldots \quad, \min\left\{2c_1, \frac{2c_2}{|\sin(2\alpha_n)|}\right\}\right\}. \tag{2}$$

If $c_1 >> c_2$, then simply

$$Rc = 2c_2 \cdot \min\left\{\frac{1}{|\sin(2\alpha_1)|}, \quad \ldots \quad, \frac{1}{|\sin(2\alpha_n)|}\right\} \tag{3}$$

for n independent random variables $\alpha_i(\omega)$. Thus R_c, which is to be found, is also a random variable. The complete probabilistic information about $R_c(\omega)$ is presented by its cpd F_n. Only the limiting case of $c_1 \to +\infty$ is analysed hereafter.

3. GENERAL SOLUTION

Let the random strength $R_c(\omega)$ in (3) have a cpd denoted as $\mathsf{F}_n(r) = \mathrm{P}\{\omega: R_c(\omega) < r\}$, which is unknown. Consider also its dimensionless equivalent $\mathsf{F}_n(\rho) = P\{\omega: \rho c(\omega) < \rho\}$ by substituting of $\rho c = R_c/(2c_2) \geq 1$ and $\rho = r/(2c_2) \geq 1$. The following solution can be derived (Appendix 2) using a variable $\alpha_\rho = \arcsin(1/\rho)/2 \leq \pi/4$

$$\mathsf{F}_n(\rho) = 1 - [F(\alpha_\rho) + F(\pi/2 + \alpha_\rho) - F(\pi/2 - \alpha_\rho) + 1 - F(\pi - \alpha_\rho)]^{\,n}. \tag{4}$$

The solution (4) holds true for $\rho \geq 1$, i.e. for $r \geq 2c_2$. There is $\mathsf{F}_n(\rho) = 0$ for $\rho < 1$ since the strength of any fissured sample cannot be less than $2c_2$.

The fissure distribution $F(\alpha)$assumed is a functional parameter of the structured materials under consideration and plays a pivotal role in solution (4). Two special classes of this function are of interest: a random isotropy and random anisotropies.

4. TOTAL CHAOS OF THE STRUCTURE – RANDOM ISOTROPY

Every sample, like the ones with n = 8 discontinuity lines in figure 1, reveals a kind of anisotropy – as far as deterministic approach is applied. Clearly, another sample of a real geomaterial, even if also composed of n = 8 fissures, is different in the sense of anisotropy. This remark is responsible for some difficulties in deterministic models, in contrast to the probabilistic one.

Consider a sample, for which the angles α_i have the same uniform distribution in [0, π]. Hence, it is assumed that orientation of each fissure is equally probable. This case will be called as "total chaos", because – generally speaking – no tendencies in fissure orientation can be observed. The sample structure is like the one in figure 1a. The term "random isotropy" has also a justification, though in a mean sense only.

Since the probability density function $f(\alpha) = dF/d\alpha$ is constant in the interval [0, π], so F is a locally linear function

$$F(a) = \mathrm{P}\{\alpha < a\} = \begin{cases} 0 & \text{for} \quad a \le 0, \\ \dfrac{1}{\pi}\cdot a & \text{for} \quad 0 < a \le \pi, \\ 1 & \text{for} \quad a > \pi. \end{cases} \tag{5}$$

For this special case solution (4) can be expressed simply as

$$\mathsf{F}_n(\rho) = 1 - \left[\frac{4\cdot\alpha_\rho}{\pi}\right]^n = 1 - \left[\frac{2\cdot arc\sin(1/\rho)}{\pi}\right]^n. \tag{6}$$

In particular, the at least 50% increase of the sample strength (i.e. $R_c \ge 3c_2$ or $\rho = 1.5$) is guaranteed with the probability 1 – 0.78 = 0.22 for n = 2, but only with the probability 1 – 0.97 = 0.03 for n = 5. This example is illustrated in figure 2.

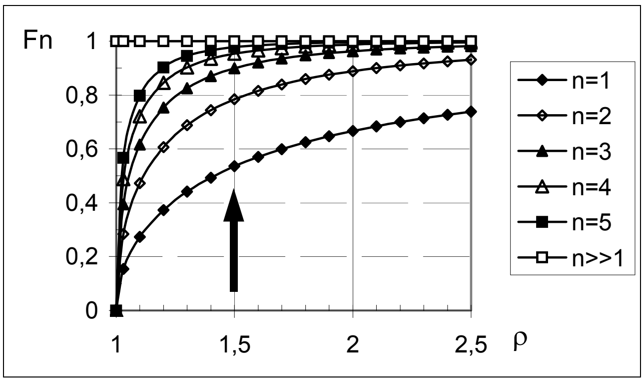

Fig. 2. Random isotropy. Cumulative probability distribution of the random strength R_c. Plots of the cpd $\mathsf{F}_n(\rho) = \mathrm{P}\{R_c/2c_2 < \rho\}$ versus dimensionless strength $\rho = r/2c_2$ for $n \ge 1$

Generally speaking, the strength R_c concentrates here not far from the minimal value possible: $\min\{R_c\} = 2c_2$. The related problems, concerning the pdf of R_c and its probabilistic moments (average value, standard deviation), are addressed in [2].

Remark: note a fast (exponential) strength decrease if the number n of fissures increases.

For any given ρ and $n \to +\infty$ the distribution F_n tends to the Heaviside function, discontinuous at $\rho = 1$. Therefore, for very large n the minimal value of R_c is reached, i.e. $R_c = \min\{R_c\} = 2c_2$ with the probability one. On the other hand, for finite n the value of R_c can be very large, though with a negligible probability. This is a consequence of the simplifying assumption that $c_1 \to +\infty$.

A correspondence to the standard homogenization technique [3] can be addressed here, especially if using a mean value of the random strength R_c, instead of the full cpd solution (6).

5. RANDOM ANISOTROPY

There exist different fissured anisotropic geomaterials described by the distributions $F(\alpha)$ – in contrast to the only one isotropy defined in (5). Note that also some discontinuous combinations of the Heaviside functions can appear in $F(\alpha)$, if some positions of fissures are fixed (the Dirac-delta pdf components).

When focusing on continuous distributions $F(\alpha)$, a class of bimodal pdfs seems to be useful for fissured geomaterials. Indeed, one can often observe that the fissure orientations appear close to two angles, say $a_1 = 30^\circ$ ($\pi/6$) and $a_2 = 120^\circ$ ($2\pi/3$) (figure 1b). Bearing this in mind, consider two fixed angles $0 \le a_1 < a_2 \le \pi$ and the following pdf $f(\alpha) = dF/d\alpha$ concentrated in the interval $0 \le \alpha \le \pi$

$$f(\alpha) = A + B \cdot \cos\left(2\pi \frac{\alpha - a_1}{a_2 - a_1}\right) \ge 0 . \tag{7}$$

The examples of such bimodal pdfs (7) are plotted in figure 3 for the domain $0^\circ \le \alpha \le 180^\circ$.

Two modal values of the pdf, i.e. both scalars a_1 and a_2, correspond to the standard modelling of sample-fabric orientation [3], whereas the parameters A, B are to model the range of random deviations of the fabric.

Integration of the assumed pdf (7) yields the following cpd $F(a) = \mathrm{P}\{\alpha < a\}$

$$F(a) = \begin{cases} 0 & \text{for} \quad a \le 0, \\ A \cdot a + B \cdot \dfrac{a_2 - a_1}{2\pi} \cdot \left[\sin\left(2\pi \dfrac{a - a_1}{a_2 - a_1}\right) + \sin\left(2\pi \dfrac{a_1}{a_2 - a_1}\right)\right] & \text{for} \quad 0 \le a \le \pi, \\ 1 & \text{for} \quad a \ge \pi. \end{cases} \tag{8}$$

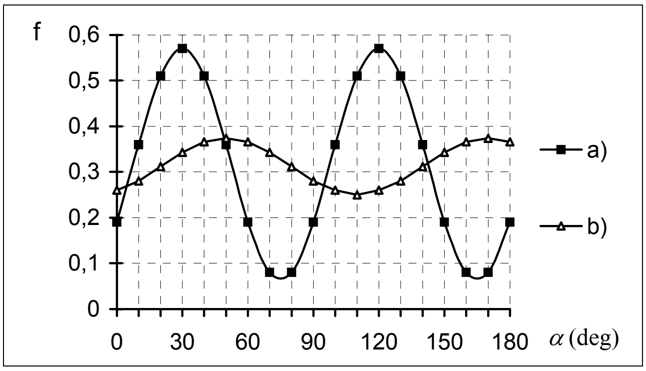

Fig. 3. Modelling of random bimodal anisotropies:
a) strong anisotropy – plot of the pdf in equation (8) for $a_1 = 30^{\circ}$, $a_2 = 120^{\circ}$ and $A = 1/\pi$, $B = 0.8A$,
b) weak anisotropy – plot of the pdf in equation (8) for $a_1 = 50^{\circ}$, $a_2 = 170^{\circ}$ and $A = 0.312$, $B = 0.2A$

Generally, the constants A and B are not independent, because $F(a)$ is a continuous increasing function, usually $A \geq B \geq 0$ and always $F(\pi) = 1$. Note that (8) reduces to (5), if $A = 1/\pi$ and $B = 0$.

Now, consider the opposite limiting cases of $A = 1/\pi = B$. For this case, the general solution (4) uses expression (8). Selected results, analogous to the ones in figure 2, are illustrated in figure 4 for $a_1 = 30^{\circ}$ and $a_2 = 120^{\circ}$.

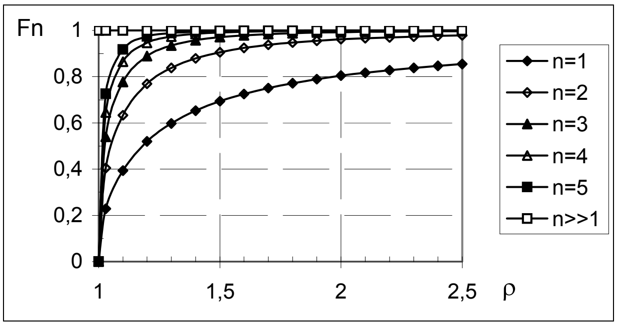

Fig. 4. A random anisotropy. Cumulative probability distribution of the random strength R_c.
Plots of the cpd $\mathsf{F}_n(\rho) = \mathrm{P}\{R_c/2c_2 < \rho\}$ versus dimensionless strength $\rho = r/2c_2$
for $a_1 = 30^{\circ}$, $a_2 = 120^{\circ}$ and $n \geq 1$

The probabilistic modelling has its limitations and specific features. The case of $n = 1$ means that only one fissure occurs in a sample, so it is either left- or right-oriented (figure 1b), and both cases occur with the same probability ½. Most probably, fissure orientation is close to $a_1 = 30^{\circ}$ or $a_2 = 120^{\circ}$ due to the shape of bimodal distribution. For $n = 2$ the most probable orientations are the same, but this does not necessarily mean that exactly one fissure is left- and the other one is right-oriented. Such a situation takes place with probability ½, whereas two left- and two right-oriented fissures can happen with the probability ¼, etc.

Another numerical study focuses on a rotational effect. Different values of the mode a_1 are considered but always $A = B = 1/\pi$ and $a_2 = a_1 + 90^o$ ($a_1 + \pi/2$) (figure 5). Thick curve denoted as (*i*) refers to the total chaos solution, i.e. random isotropy.

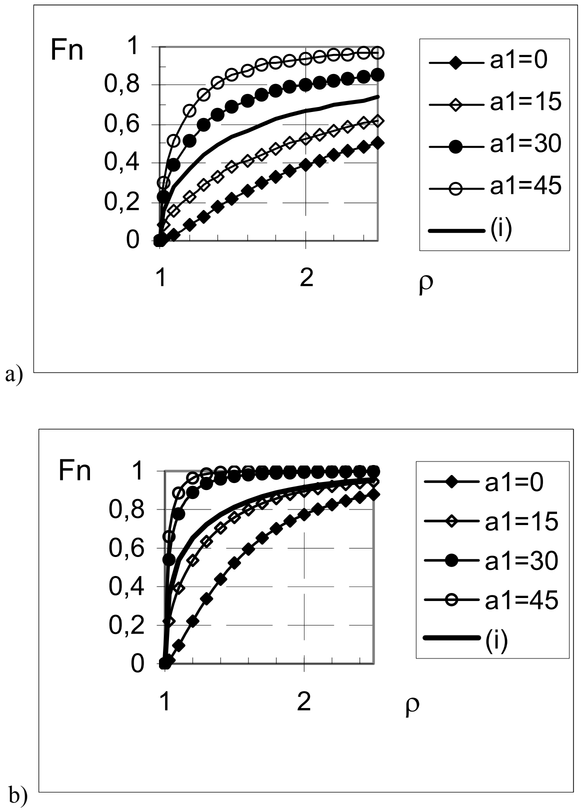

Fig. 5. Random strength R_c. Rotational effect against the modal value a_1 for $a_2 = a_1 + 90^o$.
Plots of the cpd $\mathsf{F}_n(\rho) = \mathrm{P}\{R_c/2c_2 < \rho\}$ versus dimensionless strength $\rho = r/2c_2$:
a) for $n = 1$, b) $n = 3$

6. CONCLUSIONS

1. The model presented can reflect fundamental properties of a simplified fissured material, such as the influence of the number of fissures and the fissure position (angular orientation).

2. The application of the Tresca model to the filling material has rather a technical justification, not the physical one. Such an assumption can considerably simplify derivations and presentation of results, because the Tresca failure criterion does not depend on the first invariant of the stress tensor. Thus the simplest unconfined 1D compression tests are representative of the strength analysis.

3. Simple closed-form solutions are derived based on elementary calculus of probability (Appendix 1, Appendix 2). The correspondence to the classical distribution of minimum-statistics should be addressed here [1].

4. The number of fissures n is a pivotal parameter, as far as randomly isotropic fissure orientation is considered. And opposite, for fissure distributions that are bimodally concentrated in the interval $[0, \pi]$, this dependence on n can be less significant. In a limit passage, if all fissures become deterministically parallel, the solution does not depend on n, because the power n does not change the Heaviside function. But the solution still strongly depends on the fissure orientation. This finding coincides with the behaviour of deterministic periodic structures that describe multiple sedimentary layers or composites.

5. The assumption that $c_1 = +\infty$ has some limitations. If associated with a continuous spectrum of values of the angle α, it yields a singularity, which results in unbounded interval for the values R_c. Obviously, very large strength values R_c cannot be excluded, though they are negligibly probable. Also short fissures or microcracks, being completely included within the interior of the sample, are beyond the scope of the model if $c_1 = +\infty$.

6. A generalised solution for $0 < c_2 < c_1 < +\infty$ (so $c_1 \sim c_2$) can be derived in a quite similar way. This case leads to a more realistic modelling, since the values of the strength R_c are always bounded for real materials. Details will be presented elsewhere [2].

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APPENDIX 1

Recall three facts.

Fact 1: for every random event A and its complementary random event $A^c = \Omega - A$

$$P\{A^c\} = 1 - P\{A\}\,.$$

Fact 2: for every set of numbers or functions $\{X_i\}$

$$[\min\{X_1, X_2, ..., X_n\} \geq \rho] \text{ if and only if } [X_1 \geq \rho \text{ and } X_2 \geq \rho \text{ ... and } X_n \geq \rho].$$

Fact 3: for independent random events A_i

$$\mathrm{P}\{A_1 \cap A_2 \cap ... \cap A_n\} = \mathrm{P}\{A_1\} \cdot \mathrm{P}\{A_2\} \cdot ... \cdot \mathrm{P}\{A_n\} \cdot$$

APPENDIX 2

Having in mind the Appendix 1 and making use of (3), the following sequence of equalities is true:

$$\mathsf{F}_n(\rho) = \mathrm{P}\{\omega : R_c(\omega) < r\} = \mathrm{P}\{\omega : \rho_c(\omega) < \rho\} = \mathrm{P}\{\omega : \min\{1/|\sin(2\alpha_1)|, ... ,1/|\sin(2\alpha_n)|\} < \rho\}$$

$$\overset{(F.1)}{=} 1 - \mathrm{P}\{\omega :\ \min\{1/|\sin(2\alpha_1)|, ... ,1/|\sin(2\alpha_n)|\} \geq \rho\}$$

$$\overset{(F.2)}{=} 1 - \mathrm{P}\{\omega :\ 1/|\sin(2\alpha_1)| > \rho \text{ and ... and } 1/|\sin(2\alpha_n)| \geq \rho\} \overset{(F.3)}{=} 1 - \prod_{i=1}^{n} \mathrm{P}\{\omega :\ 1/|\sin(2\alpha_i)| \geq \rho\}\ .$$

But – by assumption – the probability distributions of the orientation angles α do not depend on the parameter i. Thus

$$\mathsf{F}_n(\rho) = 1 - \prod_{i=1}^{n} \mathrm{P}\{\omega :\ 1/|\sin(2\alpha_i)| \geq \rho\} = 1 - [\mathrm{P}\{\omega :\ 1/|\sin(2\alpha)| \geq \rho\}]^n = 1 - [\mathrm{P}\{\omega : |\sin(2\alpha)| \leq 1/\rho\}]^n .$$

By inverting locally the sine function $|\sin(2\alpha)|$ for $0 < \alpha < \pi$ we arrive at

$$\mathsf{F}_n(\rho) = 1 - [\mathrm{P}\{\omega : 0 \leq \alpha \leq \alpha_\rho\} + \mathrm{P}\{\omega : \pi/2 - \alpha_\rho \leq \alpha \leq \pi/2 + \alpha_\rho\} + \mathrm{P}\{\omega : \pi - \alpha_\rho \leq \alpha \leq \pi\}]^n ,$$

where $\alpha_\rho = \arcsin(1/\rho)/2 < \pi/4$.

Making use of the given continuous cumulative probability distribution $F(\alpha)$ results in

$$\mathrm{P}\{\omega : \alpha^* \leq \alpha \leq \alpha_*\} = \mathrm{P}\{\omega : \alpha^* \leq \alpha < \alpha_*\} = F(\alpha_*) - F(\alpha^*)\ , \quad F(\pi) = 1\ , \quad F(0) = 0,$$

so finally

$$\mathsf{F}_n(\rho) = 1 - [F(\alpha_\rho) - 0 + F(\pi/2 + \alpha_\rho) - F(\pi/2 - \alpha_\rho) + 1 - F(\pi - \alpha_\rho)]^n .$$

The derivation is completed.