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# DEPENDENCE OF DYNAMIC MODULI ON THE MEAN STRESS AND VOID RATIO IN THE LIGHT OF EXPERIMENTAL DATA

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**Abstract**: The dependence of dynamic shear and bulk moduli on void ratio and mean effective stress is analysed. After a short discussion, a form of functional dependence proposed by Stoll has been chosen as the most promising. Using experimental data for three kinds of the Monterey sand and the Warsaw Pliocene clay (taken from literature) the regression and multiregression analyses have been carried out in order to identify the value of material constants appearing in the expressions discussed. The analysis has firmly confirmed (high correlations and very good values of other statistics) that the general relationship proposed by Stoll has a character of a physical law. The exponent *n* in this general model is relatively nonsensitive to a kind of soil, but the parameter related to the void ratio is strongly dependent on it. Both these parameters are different for bulk and shear moduli. Some other conclusions concerning the nature of this dependence have been also formulated.

## 1. FORMULATION OF THE PROBLEM

The values of dynamic shear and bulk moduli are fundamentals of any reliable constitutive model of soils. Its correct approximation allows us to analyse wave propagation, vibrations and design of foundations for dynamic loading. Even computation of static settlement needs the realistic estimation of shear modulus. The values of dynamical moduli are crucial for determination of leading, pseudoelastic terms in the rate-type constitutive equations.

Nowadays, the most common expression linking the shear modulus with soils parameters and mean stress is that attributed to HARDIN and BLACK [2]:

$$G = Af(e)\sigma_0'^{1/2}. \qquad (1)$$

In expression (1), $G$ stands for the shear modulus, $A$ is a material constant, $f(e)$ denotes a function of void ratio $e$, while $\sigma_0'$ is mean effective stress. The influence of the two most important factors: the void ratio $e$ and mean effective stress $\sigma_0'$ on $G$ is given in (1). The coefficient $A$ is specific to a kind of soil. The value $n = 0.5$ of the power of the mean effective stress in (1) has been assumed *a priori* based on the Hertz research on the stress distribution in bodies in contact and on the research of Mindlin, Cattaneo, Duffy or Deresiewicz. The corresponding references are listed in the works by KISIEL and LYSIK [5] or STOLL [7]. These results are valid for the strains whose values vary in the range of $10^{-5}$–$10^{-7}$.

The form of the function $f(e)$ can be justified by a hypothesis that a measured velocity of shear wave divided by square root of mean effective stress is linear with respect to void ratio:

$$\frac{v_s}{\sigma_0'^{1/4}} = a - be\,. \tag{2}$$

In the above expression, $v_s$ stands for shear wave velocity, while $a$ and $b$ are constant parameters. Taking into account the well-known definition of the shear modulus we obtain immediately:

$$G = \frac{\rho_s b(c-e)^2}{1+e}\sigma_0'^{1/2}\,. \tag{3}$$

In the above expression, $\rho_s$ denotes a specific density, and $c$ is a new constant equal to $a/b$. Formula (3) is the most known form of the expression (1) commonly used in soil dynamics since the sixties of the twentieth century. In 1988, STOLL [7] published different form of the expression for shear modulus. His formula is based on a large number of samples of marine sediments. The number of measurements that supported his formula was about 100 times higher than that of Black and Hardy. Using multiple linear regression method, Stoll proposed the following formula for the shear modulus:

$$G = a\exp(-be)\sigma_0'^{n}\,. \tag{4}$$

Now, three material parameters are to be defined: $a$ that depends on material properties, $b$ that determines a quantitative influence of the void ratio, and $n$ which according to the Hertz law must be much lower than 1.0.

After abridging the variables equation (4) is represented by the following linear form:

$$\ln G = \ln a - be + n\ln\sigma_0'\,. \tag{5}$$

Stoll proposed $b$ = 1.54 and $n$ = 0.448 as common parameters for most of the soils. Also Black and Hardin argue that the exponent $n$ is common to cohesive soils. However, other experimental results obtained without eliminating the influence of void ratio show that $n$ varies between 0.7 and 1.0. This is in the case of the so-called "initial" Young moduli $E_{oi}$ used in the zero-order hypoelastic models of DUNCAN and CHANG [1]. The values of $n$ for glacial tills from Łódź and Weald London clays [3], [4] are equal to 0.69 and 1.0, respectively.

The purpose of the paper was to explain these differences. Relation (6) representing laboratory tests and relation (7) for *in situ* observations make a pood starting point for this:

$$\log G = \log a - be + n\log\sigma_0'\,, \tag{6}$$

$$\log G = \log a_1 - b_1 e + n \log z \ , \tag{7}$$

where $z$ is the depth of the layer considered. Rewriting the above law we arrive at:

$$G = 10^a 10^{be} \sigma_0'^n \ . \tag{8}$$

Along with formulae (6) and (7) we will examine the following additional form of the equation for shear modulus, excluding its explicit dependence on $e$:

$$\log G = \log a_2 + n_1 \log \sigma_0' \ , \tag{9}$$

$$\log G = \log a_3 + n_1 \log z \ . \tag{10}$$

Also the following possible sub-correlations will be taken into account:

$$\log G = \log d_1 - d_2 e \ , \tag{11}$$

$$e = c_0 - c_c \log \sigma_0' \ , \tag{12}$$

$$e = c_1 - c_2 \log z \ . \tag{13}$$

In other words: the purpose of the paper is to verify the "geometrical" nature of relations (6) and (7). Do they represent plane surfaces or a line in the space occupied by the axes $\log G$, $e$, $\log \sigma_0'$? If relations (9)–(12) are valid, it will be possible to explain the differences in the powers of $\sigma_0'$ discussed earlier.

## 2. RESULTS OF LINEAR AND MULTI LINEAR REGRESSION ANALYSIS FOR DIFFERENT SAMPLES OF SOILS

Statistic analysis has been carried out for four kinds of soils: three Monterey sands and Warsaw Pliocene clay [6]. The first group of results consists of laboratory tests published in [8]; the second one is obtained by *in situ* CPT dynamic probe [6]. These results are shown in tables 1, 2, 3 and 4. $K$ denotes the bulk modulus.

Table 1

Experimental data for coarse Monterey sand

| No. | $K$ [kPa] | $G$ [kPa] | $e$ | $\sigma_0'$ [kPa] |
|---|---|---|---|---|
| 1 | 279600 | 48600 | 0.661 | 10.8 |
| 2 | 344200 | 65600 | 0.656 | 20.6 |
| 3 | 413500 | 85000 | 0.653 | 40.2 |
| 4 | 514100 | 115000 | 0.647 | 79.4 |
| 5 | 602300 | 172000 | 0.642 | 157.8 |
| 6 | 875000 | 237000 | 0.639 | 314.6 |
| 7 | 1084000 | 320000 | 0.637 | 628.2 |

Table 2

Experimental data for medium Monterey sand

| No. | $K$ [kPa] | $G$ [kPa] | $e$ | $\sigma'_0$ [kPa] |
|---|---|---|---|---|
| 1 | 149700 | 33100 | 0.799 | 10.8 |
| 2 | 162090 | 42000 | 0.799 | 20.6 |
| 3 | 181300 | 57100 | 0.795 | 40.2 |
| 4 | 224100 | 77300 | 0.792 | 79.4 |
| 5 | 280300 | 108000 | 0.789 | 157.8 |
| 6 | 389600 | 152000 | 0.786 | 314.6 |
| 7 | 529600 | 214000 | 0.783 | 628.2 |

Table 3

Experimental data for fine Monterey sand

| No. | $K$ [kPa] | $G$ [kPa] | $e$ | $\sigma'_0$ [kPa] |
|---|---|---|---|---|
| 1 | 101800 | 31200 | 0.811 | 10.8 |
| 2 | 120800 | 37000 | 0.811 | 20.6 |
| 3 | 150600 | 49000 | 0.808 | 40.2 |
| 4 | 187800 | 69100 | 0.805 | 79.4 |
| 5 | 255000 | 91100 | 0.802 | 157.8 |
| 6 | 348500 | 123000 | 0.795 | 314.6 |
| 7 | 492700 | 174000 | 0.786 | 628.2 |

Table 4

Experimental data for Warsaw Pliocene clay

| $G$ [$10^4$ kPa] | $z$ [m] | $e$ |
|---|---|---|
| 5.24 | 5.0 | 0.9 |
| 5.50 | 5.0 | 0.9 |
| 5.70 | 6.0 | 0.9 |
| 5.50 | 6.0 | 0.9 |
| 5.99 | 7.0 | 0.69 |
| 6.49 | 7.0 | 0.69 |
| 6.13 | 8.0 | 0.69 |
| 6.64 | 8.0 | 0.69 |
| 7.82 | 9.0 | 0.56 |
| 8.15 | 9.0 | 0.56 |
| 9.89 | 10.0 | 0.56 |

It is known that there exist some partial correlations between independent variables of the problem as well as between dependent variables. To discover them expressions (6) and (7) and (9)–(13) have been statistically processed based on the data collected in tables 1–4.

The analysis that has been carried out consists of the following operations: first, we have taken logarithms of booths: mean effective stresses and moduli from tables 1–3. In the case of Warsaw Pliocene clay (table 4), we have used the logarithm of depths $z$ instead of the logarithm of stress. Next we have applied the classical multiregression and regression procedures to find the coefficients in the above-mentioned formulas. As a result of this analysis, formulae (6), (7) and (9)–(13) can be substituted for the following expressions, in which unknown coefficients are identified.

For the coarse Monterey sand (table 1) we have obtained the following relations:

$$\lg K = 5.934 - 1.2491e + 0.314 \lg(\sigma_0 / l), \tag{14}$$

$$\lg G = 4.8258 - 0.9401e + 0.457 \lg(\sigma_0 / l). \tag{15}$$

These abridged forms are naturally associated with the expressions written in physical scale:

$$K = 8.59 \cdot 10^5 \exp(-2.876e)\,(\sigma_0 / l)^{0.314}, \tag{16}$$

$$G = 6.69 \cdot 10^4 \exp(-2.165e)\,(\sigma_0 / l)^{0.457}. \tag{17}$$

Equation (18) expresses the Casagrande law of compressibility and appears here as an auxiliary relation:

$$e = 0.6749 - 0.0416 \lg(\sigma_0 / l). \tag{18}$$

We noticed that insertion of the compressibility law (18) into relations (14) and (15) results in the following shortened formulae for the moduli $G$ and $K$:

$$K = 1.23 \cdot 10^5 (\sigma_0 / l)^{0.332}, \tag{19}$$

$$G = 1.55 \cdot 10^4 (\sigma_0 / l)^{0.470}. \tag{20}$$

We underline that expressions (19) and (20) are true in the case, where variation of the void ratio is caused only by effective mean stress.

For the medium Monterey sand (table 2) we have obtained the following relations:

$$\lg K = 6.647 - 2.293e + 0.295 \lg(\sigma_0 / l), \tag{21}$$

$$\lg G = 5.720 - 2.130e + 0.445 \lg(\sigma_0 / l). \tag{22}$$

The physical forms corresponding to the above equations are the following:

$$K = 4.44 \cdot 10^6 \exp(-5.28e)\,(\sigma_0 / l)^{0.295}, \tag{23}$$

$$G = 5.25 \cdot 10^5 \exp(-4.905e)\,(\sigma_0 / l)^{0.457}. \tag{24}$$

Equation (25) expresses the Casagrande law of compressibility for the medium Monterey sand:

$$e = 0.810 - 0.0097\lg(\sigma_0 / l). \tag{25}$$

Insertion of the compressibility law (25) into relations (23) and (24) results in the following shortened formulae for the moduli $G$ and $K$:

$$K = 6.20 \cdot 10^4 (\sigma_0 / l)^{0.315}, \tag{26}$$

$$G = 1.05 \cdot 10^4 (\sigma_0 / l)^{0.464}. \tag{27}$$

We underline that expressions (26) and (27) are true in the case, where variation of the void ratio is caused only by effective mean stress.

For the fine Monterey sand (table 3) the following relations take place:

$$\lg K = 12.17 - 9.2e + 0.277\lg(\sigma_0 / l), \tag{28}$$

$$\lg G = 7.569 - 4.30e + 0.379\lg(\sigma_0 / l). \tag{29}$$

Writing (28) and (29) in a physical scale we obtain:

$$K = 1.47 \cdot 10^{11} \exp(-21.18e)\,(\sigma_0 / l)^{0.277}, \tag{30}$$

$$G = 3.71 \cdot 10^{7} \exp(-9.901e)\,(\sigma_0 / l)^{0.379}. \tag{31}$$

As before, an auxiliary relation between the void ratio and the hydrostatic pressure can be written down:

$$e = 0.8287 - 0.0137\lg(\sigma_0 / l). \tag{32}$$

The following shortened formulae for the moduli $G$ and $K$ result from insertion of (32) into (30) and (31):

$$K = 3.72 \cdot 10^4 (\sigma_0 / l)^{0.389}, \tag{33}$$

$$G = 1.05 \cdot 10^4 (\sigma_0 / l)^{0.431}. \tag{34}$$

Expressions (33) and (34) are valid in the case, where variation of the void ratio is due to effective mean stress only.

Similar results, concerning in this case (for the obvious reason) only the shear dynamic modulus, are obtained for the Warsaw Pliocene clay (table 4):

$$\lg G = 4.5637 - 0.2079e + \lg(z / l). \tag{35}$$

Corresponding expression for shear dynamic modulus:

$$G = 3.66 \cdot 10^4 \exp(-0.4787e)\,(z / l)^{0.473}. \tag{36}$$

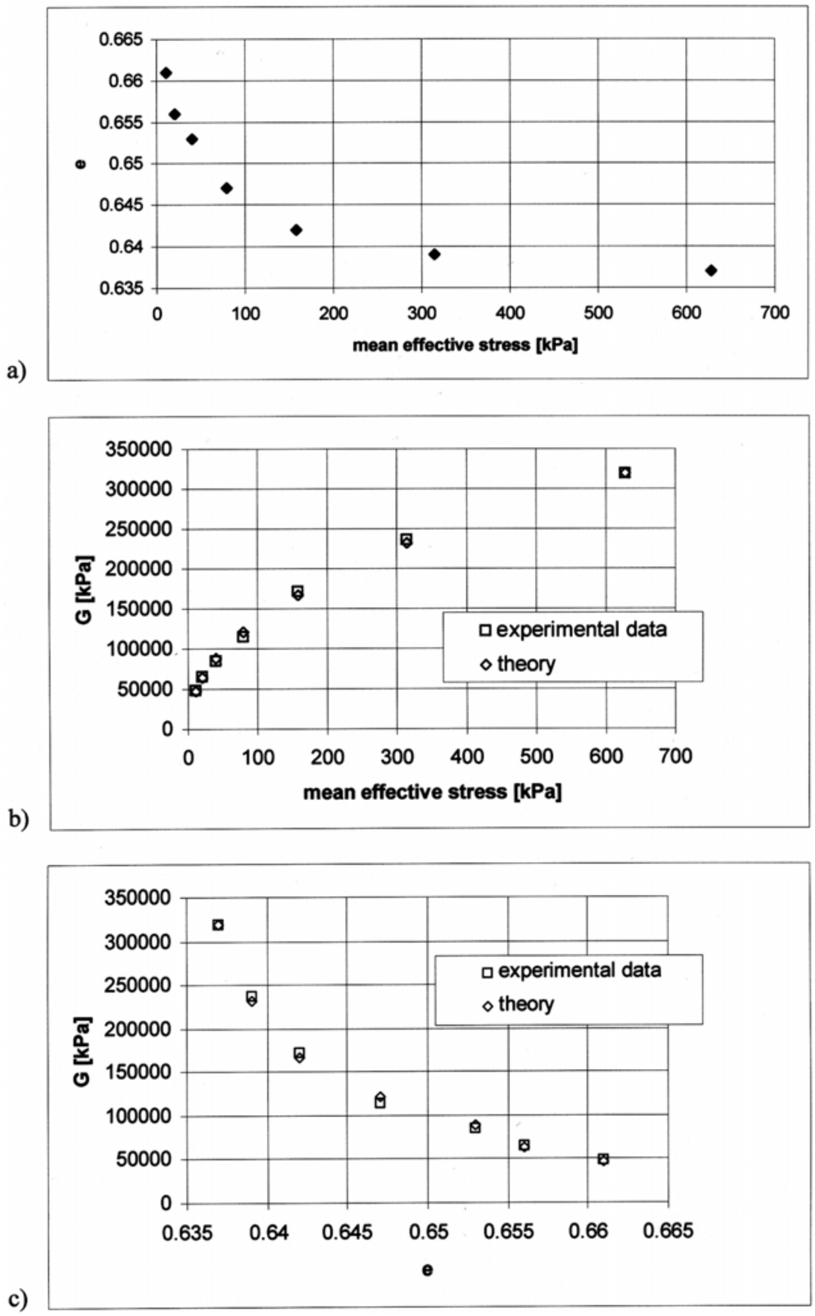



Fig. 1. Projection of the curve defining the dynamic shear modulus *G* for the coarse Monterey sand on the planes of the system of coordinates: a) on the plane: void ratio – mean effective stress, b) on the plane: mean effective stress – *G*, c) on the plane: void ratio – *G*

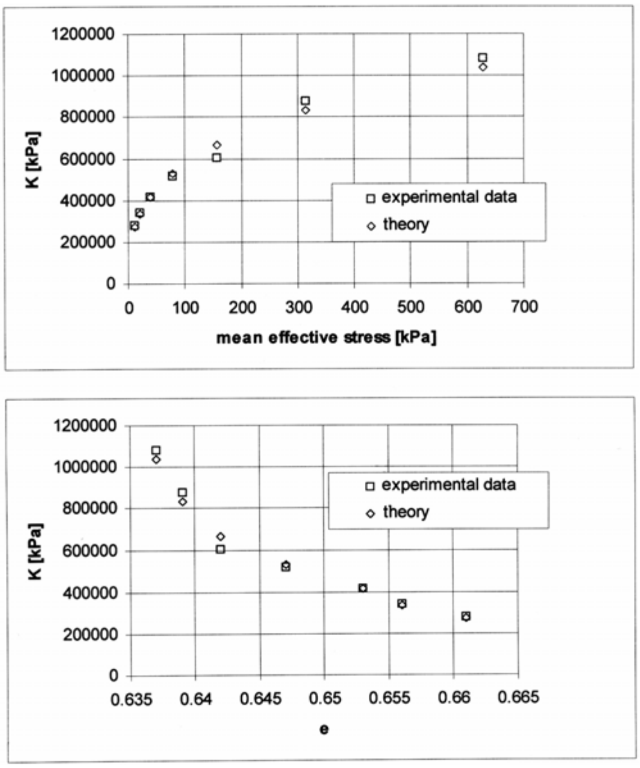
The following auxiliary relation represents an equivalent form of the Casagrande law of compressibility:

$$e = 1.861 - 1.3281\lg(z/l)\,. \tag{37}$$

A shortened formula (38) for $G$ is valid in the case, where variation of the void ratio is caused only by effective mean stress:

$$G = 1.49 \cdot 10^4 z^{0.753}\,. \tag{38}$$

During statistical analysis we have controlled the coefficient of determination $r^2$ and the level of significance $F$ (ratio of variances). The coefficient of determination $r^2$ varied in the range of 83.0%–99.8%. The parameter $F$ reflects practically deterministic character of the relations identified.




a)

b)

Fig. 2. Projection of the curve defining the dynamic bulk modulus $K$ for the coarse Monterey sand on the planes of the system of coordinates: a) on the plane: mean effective stress – $K$, b) on the plane: void ratio – $K$

For a graphical illustration we have chosen the results obtained for the coarse Monterey sand and the Warsaw Pliocene clay. In figure 1 and figure 3, the projections of relation (4) on the planes of the coordinate system are plotted for the coefficients identified by the regression and multiregression analyses. Theoretical graphs are compared with experimental observation taken from tables 1 and 4. The same concerns the bulk dynamic modulus in figure 2.

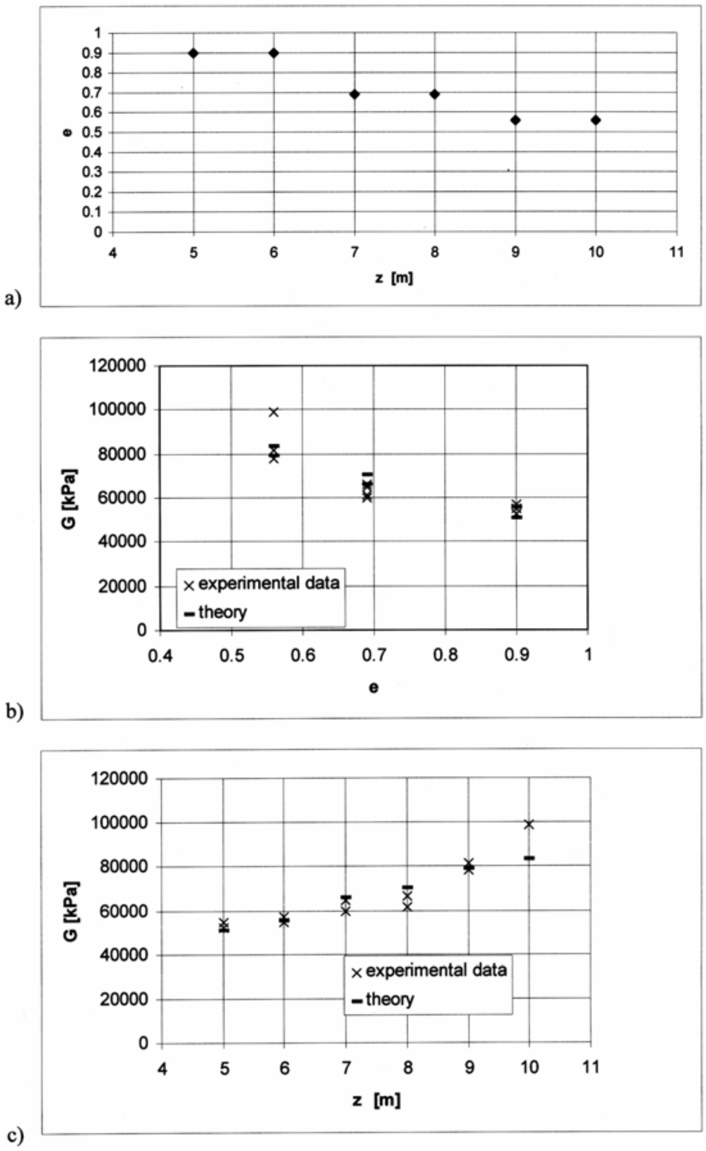



Fig. 3. Projection of the curve defining the dynamic shear modulus *G* for the Warsaw Pliocene clay on the planes of the system of coordinates: a) on the plane: void ratio – depth of the layer, b) on the plane: void ratio – *G*, c) on the plane: depth of the layer – *G*

## 3. CONCLUSIONS

It can be concluded that relation $G = f(e, \sigma_0')$ has a geometrical meaning of a curve and has not a surface in the space occupied by the coordinate system ($G$, $e$, $\sigma_0'$).

In the case where a loading path is characterized by a variation of the mean stress only, the Casagrande law of compressibility allows us to eliminate *e* from the formulae. Thus the mean stress plays a double role in determining the value of the dynamic moduli. In a general case of loading, a full form of the expression should be accounted for. The exponent *n* exhibits a character of a true physical constant. This exponent for the bulk modulus *K* corresponds closely to the Hertz theory, and for shear moduli it is practically equal to 0.5, as it has been stated before by many authors. In the light of our studies, the parameter *b* is strongly dependent on granulometric curve of soil. This means that the opinion and observation of STOLL published in [7] are not justified.

The multi-linear regression analysis shows undoubtedly (high correlations and very good values of other statistics) that the following conclusions can be formulated:

- A general relationship (4) proposed first by Stoll has a character of a physical law.
- The simplified relations (without void ratio) are valid for particular paths of loading (increasing mean stress only).
- The exponent *n* in a general model is relatively non-sensitive to a kind of soil.
- The parameter *b* related to the void ratio is strongly dependent on a kind of soil.
- Both parameters *n* and *b* are different for the bulk and shear moduli.
- The Casagrande law reveals the difference between the general and simplified models.
- The Hertz law is not verified if its simplified form is used, especially for cohesive soils (the exponent *n* in these particular laws does not obey the Hertz law).

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