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THEORETICAL ANALYSIS OF PARTIALLY COMPOSITE BEAMS WITH CHOSEN EXAMPLES OF CONNECTOR DENSITY

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Abstract: When designing composite beams it is important to assure an appropriate stiffness of concrete-steel connection to achieve the strength required and the useful parameters. Many factors influence the features of composite beam. One of them is a function of connectors' density. Usually designers apply a uniform distribution of connectors along beam length, although it hardly ever is an optimal solution. In this article, you can find the solutions of displacements for simply supported composite beam with three different ways of connectors' spacing along its span. The results are presented as the ratios of the corresponding deflections of the equivalent fully composite beams.

Streszczenie: Podczas projektowania belek zespolonych ważne jest zapewnienie odpowiedniej sztywności w połączeniu betonu i stali, co zapewnia wymaganą sztywność i odpowiednie parametry użytkowe. Na właściwości belek zespolonych ma wpływ wiele czynników. Jednym z nich jest funkcja rozkładu gęstości podłużnej łączników. Zwykle projektanci stosują równomierny rozkład łączników zespalających wzdłuż osi belki, choć rzadko jest to rozwiązanie optymalne. W niniejszym artykule można znaleźć rozwiązania przemieszczeń belki zespolonej o schemacie statycznym wolnopodpartym dla trzech różnych sposobów rozmieszczenia łączników zespalających na długości przęsła. Wyniki przedstawiono jako współczynniki odpowiadających przemieszczeń dla belek z całkowitym (sztywnym) zespoleniem.

Резюме: Во время проектирования соединенных балок важным является обеспечение соответствующей жесткости соединения бетона и стали для достижения их требуемой жесткости и эксплуатационных параметров. На свойства соединенных балок влияют многие факторы. Одним из них является функция распределения продльной плотности соединению Обычно проектировщики применяют равномерное распределение связывающих соединений вдоль оси балки, несмотря на то, что это решение редко является оптимальным. В настоящей статье можно найти решения смещений соединенной балки статической, свободнокрепленной схемы для трех разных способов распределения соединяющих элементов на всей длине пролетаю Результаты были представлены в виде коэффициентов соответствующих смещений для балок с полным (жестким) соединением.

1. INTRODUCTION

In recent years, constructors have commonly applied composite structures. The most popular elements of this type became steel–concrete beams. In such supporting elements, a reinforced concrete and steel (two different materials), joined together by special connectors, co-operate in load bearing.

Based on the experience gained it is known that in an optimal connection between steel and concrete, fewer number of connectors should be used than in fully composed beams [1], but simultaneously connectors must be so flexible as to assure a certain slip in the steel–concrete surface. Therefore current practice [2], [3], [4] permits composite beams to be designed with partial shear connection.

Practical methods of calculation of composite girders with partial connection were developed. Displacements of beams with partial connection are shown at some charts [5], [6], [7], [8] representing many static schemes and ways of loading. The displacements at charts are identified as dimensionless ratios being the quotients of displacements of beam and displacements of an equivalent beam being fully jointed. For the sake of comparison of the results achieved, similar characteristics were used in this article.

In all the papers mentioned above, calculations are based on the assumption that the convectors are equally spaced along a span. It is also commonly known that for simply supported beams a shear force in steel–concrete surface is concentrated towards supports and decreases to zero at midspan. An increase in the efficiency of connection may be achieved by non-linear connectors' density function which assures greater stiffness of connection in the supporting area.

2. SOME CHOSEN CHARACTERISTICS OF CONNECTORS' DENSITY

The method of calculating displacements of composite girders with partial connection at three different connectors' distributions is presented. In this article, we try to answer the question about the influence of connectors' density function on bending stiffness. Three options of connection are considered and presented in the form of the following mathematical formulas:

a) uniform connectors' distribution

$$g_1(x) = K_S = \frac{G}{2},$$

b) linear connectors' distribution

$$g_2(x) = \left(\frac{2K_S}{a}\right) \cdot x = \left(\frac{G}{a}\right) \cdot x,$$

c) parabolic connectors' distribution

$$g_3(x) = \left(3\frac{K_S}{a^2}\right) \cdot x^2 = \left(\frac{3}{2}\frac{G}{a^2}\right) \cdot x^2,$$

where:

K_S – the connector modulus for the case of uniformly spaced connectors,

G – the connector modulus at support,

L – is the beam length,
a – the distance from the point of maximum bending moment to support.
Functions $g_1(x)$, $g_2(x)$, $g_3(x)$ fulfil the condition:

$$\int_0^{L/2} g_1(x)dx = \int_0^{L/2} g_2(x)dx = \int_0^{L/2} g_3(x)dx,$$

where L is the beam length.

3. THEORETICAL ANALYSIS

3.1. ASSUMPTIONS

Theoretical analysis presented in this work is based on the following assumptions:

a) concrete and steel are linear elastic materials and each of them has the same elastic modulus in tension and compression,

b) the shear connection between the concrete and steel beam is continuous along the beam,

c) the extent of slip permitted by a shear connector is directly proportional to the load transmitted,

d) the concrete slab and the steel beam deflect equally at all points along the span,

e) shear connectors have the same modulus and the shear connector density varies along the span from maximum value to zero at the point of maximum bending moment,

f) the analysis presented concerns exclusively the simply supported, symmetric beams, therefore all calculations are done for x values greater than 0 ($x \geq 0$) (see figure 1).

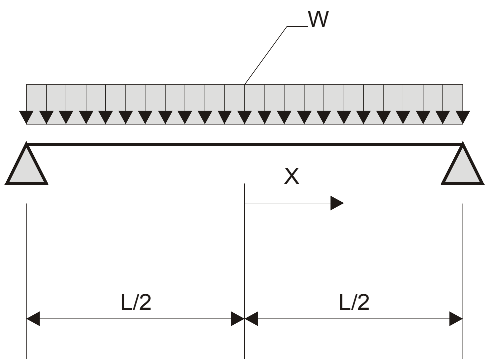


Fig. 1. Static scheme of the beam tested

3.2. STATIC SCHEME

Further analysis was conducted for simply supported, symmetric beams (figure 1).

The analysis is supplemented by the value a which makes the calculations more transparent and clear. This value represents the distance from the point of maximum bending moment to support. In the example analyzed $a = L/2$.

3.3. CALCULATIONS, ANALYTICAL PART

A parabolic function of connectors' distribution presented in [9] was taken into consideration on the basis of differential element dx (figures 2, 3).

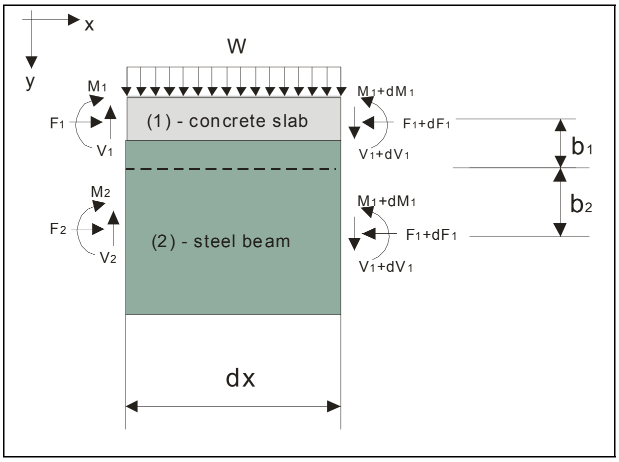


Fig. 2. Differential element of composite girder

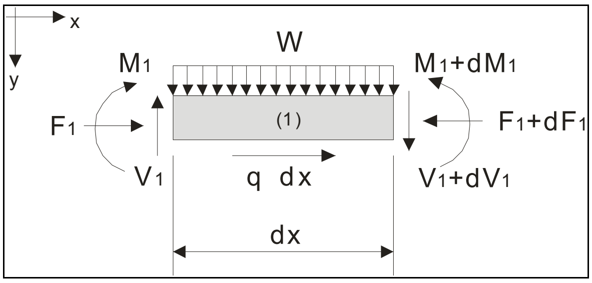


Fig. 3. Free body diagram of the upper component

Differential element of composite beam consists of concrete part (1) and steel part (2) (figure 2). The horizontal equilibrium of the upper component (figure 3) gives:

$$q = \frac{dF_1}{dx}. \tag{1}$$

The expression for the strain at interfaces of both components ((1) and (2)) may be given by:

$$e_1 = \frac{M_1}{E_1 \cdot J_1} \cdot b_1 - \frac{F_1}{E_1 \cdot A_1}, \quad e_2 = -\frac{M_2}{E_2 \cdot J_2} \cdot b_2 + \frac{F_2}{E_2 \cdot A_2}. \tag{2}$$

Relations between the elements (1) and (2) at the slip interface *s* and the strains shown above as well as their relation to shear flow are expressed as follows:

$$\frac{ds}{dx} = e_2 - e_1, \tag{3}$$

$$s = \frac{q}{g}, \tag{4}$$

$$g = g_3(x) = \left(3\frac{K_S}{a^2}\right) \cdot x^2 = \left(\frac{3}{2}\frac{G}{a^2}\right) \cdot x^2. \tag{5}$$

Inserting equation (1) into (4) and assuming that $F_1 = F_2 = F$ we arrive at:

$$s = \frac{1}{g} \cdot \frac{dF}{dx}. \tag{6}$$

In the following step, expressions for strains (2) and expression (6) are inserted into equation (3):

$$\frac{2a^2}{3G} \cdot \frac{1}{x^2}\frac{d^2F}{dx^2} - \frac{4a^2}{3G} \cdot \frac{1}{x^3}\frac{dF}{dx} = -\frac{M_2}{E_2 \cdot J_2} \cdot b_2 + \frac{F_2}{E_2 \cdot A_2} - \frac{M_1}{E_1 \cdot J_1} \cdot b_1 + \frac{F_1}{E_1 \cdot A_1}. \tag{7}$$

Having used the curvature which is given by:

$$\frac{d^2y}{dx^2} = -\frac{M}{J} + \frac{b \cdot F}{J} \tag{8}$$

and the following substitutions:

$$J = E_1A_1 + E_2A_2, \quad b = b_1 + b_2, \quad H = \frac{1}{E_1A_1} + \frac{1}{E_2A_2}$$

equation (7) may be expressed by:

$$\frac{2a^2}{3G} \cdot \frac{1}{x^2}\frac{d^2F}{dx^2} - \frac{4a^2}{3G} \cdot \frac{1}{x^3}\frac{dF}{dx} - \frac{G}{a^2} \cdot \left(H + \frac{b^2}{J}\right) \cdot F = -\frac{G \cdot b}{a^2 \cdot J} \cdot M. \tag{9}$$

Introducing the concept of transformed section (change to steel) as well as deriving expression (9) by the following substitutions:

$$G_1 = \frac{G \cdot A_m}{a^2 \cdot E_2} \cdot \left(1 + \frac{b^2}{A_m J_m}\right), \quad G_2 = \frac{G \cdot b}{a^2 \cdot E_2 J_m}, \quad J_m = J_{1t} + J_2, \quad A_m = \frac{1}{A_{1t}} + \frac{1}{A_2},$$

we arrive at

$$\frac{2a^2}{3G} \cdot \frac{1}{x^2} \frac{d^2 F}{dx^2} - \frac{4a^2}{3G} \cdot \frac{1}{x^3} \frac{dF}{dx} - G_1 \cdot F = -G_2 \cdot M, \tag{10}$$

where:

J_{1t} stands for the moment of inertia of transformed section of component (1), i.e. concrete,

A_{1t} is the transformed area of component (1), i.e. concrete.

Differential equation (10) was transformed in the way that would allow the expression for the function F to be found. Having obtained this, it was inserted into (8). After all these transformations the following expression is obtained:

$$\frac{d^2 y}{dx^2} = -\frac{M}{E_2 J_m} + \frac{G_2}{G_1} \cdot \frac{b \cdot M}{E_2 J_m} + \frac{2}{3} \cdot \frac{b}{E_2 J_m \cdot G_1} \cdot \frac{1}{x^2} \frac{d^2 F}{dx^2} - \frac{4}{3} \cdot \frac{b}{E_2 J_m \cdot G_1} \cdot \frac{1}{x^3} \frac{dF}{dx}. \tag{11}$$

In further analysis, appropriate expressions were substituted for G_1 and G_2 as well as relation (12) was adopted based on [10].

$$\frac{I}{J_m} = 1 + \frac{b^2}{A_m J_m}, \tag{12}$$

where I is the moment of inertia of transformed, fully composite section.

After making few simple transformations, equation (11) may be expressed in its final form:

$$\frac{d^2 y}{dx^2} = -\frac{M}{E_2 I} + \frac{2}{3} \cdot \frac{b}{E_2 J_m \cdot G_1} \cdot \frac{1}{x^2} \frac{d^2 F}{dx^2} - \frac{4}{3} \cdot \frac{b}{E_2 J_m \cdot G_1} \cdot \frac{1}{x^3} \frac{dF}{dx}. \tag{13}$$

A system of two differential equations, (10) and (13), was obtained as a result of the analysis presented. In these expressions, the unknowns represent the functions $F(x)$ and $y(x)$, while M is a known function of bending moment.

By analogy, similar analyses were made for two last functions of connectors' density $g_1(x)$ and $g_2(x)$. Two other systems of differential equations were derived. The functions representing three ways of steel–concrete connections are gathered below:

a) the function $g_1(x)$ – uniform distribution:

$$2 \cdot \frac{d^2F}{dx^2} - G_1 \cdot F = -G_2 \cdot M \ ,$$

$$\frac{d^2y}{dx^2} = -\frac{M}{E_2 I} + \frac{2b}{E_2 J_m \cdot G_1} \cdot \frac{d^2F}{dx^2} \ ,$$

where:

$$G_1 = \frac{G \cdot A_m}{E_2} \cdot \left(1 + \frac{b^2}{A_m J_m}\right), \quad G_2 = \frac{G \cdot b}{E_2 J_m} \ ;$$

b) the function $g_2(x)$ – linear distribution:

$$x^2 \cdot \frac{d^2F}{dx^2} - x \cdot \frac{dF}{dx} - G_1 x^3 \cdot F = -G_2 x^3 \cdot M \ ,$$

$$\frac{d^2y}{dx^2} = -\frac{M}{E_2 I} + \frac{b}{E_2 J_m \cdot G_1} \cdot \frac{1}{x} \frac{d^2F}{dx^2} - \frac{b}{E_2 J_m \cdot G_1} \cdot \frac{1}{x^2} \frac{dF}{dx} \ ,$$

where:

$$G_1 = \frac{G \cdot A_m}{a \cdot E_2} \cdot \left(1 + \frac{b^2}{A_m J_m}\right), \quad G_2 = \frac{G \cdot b}{a \cdot E_2 J_m} \ ;$$

c) the function $g_3(x)$ – parabolic distribution:

$$2x \cdot \frac{d^2F}{dx^2} - 4 \cdot \frac{dF}{dx} - 3G_1 x^3 \cdot F = -3G_2 x^3 \cdot M \ ,$$

$$\frac{d^2y}{dx^2} = -\frac{M}{E_2 I} + \frac{2}{3} \cdot \frac{b}{E_2 J_m \cdot G_1} \cdot \frac{1}{x^2} \frac{d^2F}{dx^2} - \frac{4}{3} \cdot \frac{b}{E_2 J_m \cdot G_1} \cdot \frac{1}{x^3} \frac{dF}{dx} \ ,$$

where:

$$G_1 = \frac{G \cdot A_m}{a^2 \cdot E_2} \cdot \left(1 + \frac{b^2}{A_m J_m}\right), \quad G_2 = \frac{G \cdot b}{a^2 \cdot E_2 J_m} \ .$$

The dimensions of the constants G_1 and G_2 in each of the points ((a), b), c)) are different. Depending on a further analysis carried out based on this method, it will be useful to express the constants mentioned above in the same dimension in order to make their comparison easier.

3.4. CALCULATIONS, NUMERICAL PART

Owing to the difficulties in finding an exact solution of differential equations' systems mentioned in point 3.3, a numerical analysis was applied. This differential problem was solved by using MATHEMATICA's 4.0 interface. The differences in three cases of connectors' distribution were calculated for a composite girder shown in figure 4 and described in table 1.

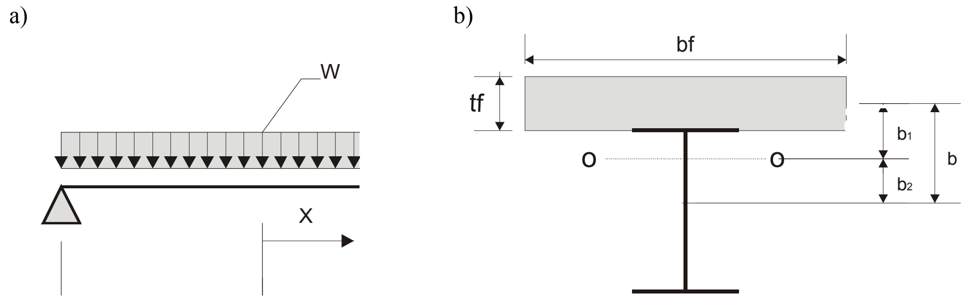


Fig. 4. The girder tested: a) static scheme, b) cross-section

Table 1

Data for numerical example

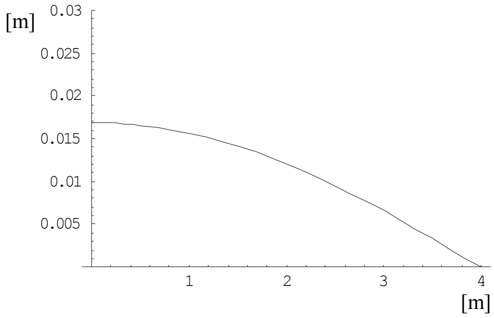
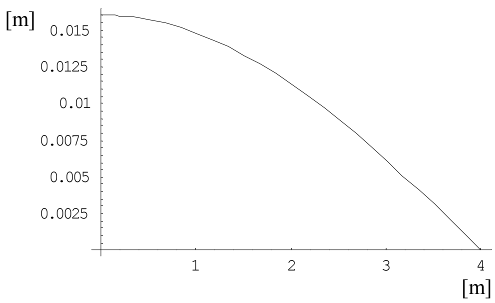
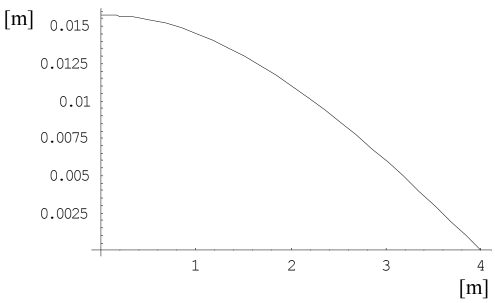
Features of concrete slab	Features of steel beam	Connection	Length and loading
$b_f = 96$ cm $t_f = 8$ cm $E_1 = 27000$ MPa	$h = 450$ mm $J_2 = 33740$ cm^4 $A_2 = 98.8$ cm^2 $E_2 = 205$ GPa	$k = 40\dfrac{\text{kN}}{\text{mm}}$	$L = 8$ m $w = 30\dfrac{\text{kN}}{\text{m}}$

A comparison of the results of calculations made for all three cases of connectors' density function based on the example of grider is given in table 2. Because of a symmetric problem these results are presented for positive values of x only. Charts of the unknown function $F(x)$ are not present in table 2. These values are necessary as an indirect step of calculations only.

In the case of the example presented, the displacement of a steel beam without any connection is $y(0) = 0.0231324$ [m], but for a fully connected composite beam it is $y(0) = 0.0112457$ [m].

Table 2

Comparison of results of numerical calculations

	Function of connectors' density	Chart of displacement $y(x)$	Value $y(x)$ at $x = 0$ [m]
1	$g_1(x) = K_S$		0.0169201
2	$g_2(x) = \left(\frac{G}{a}\right) \cdot x$		0.0160212
3	$g_3(x) = \left(\frac{3}{2}\frac{G}{a^2}\right) \cdot x^2$		0.0157385

Comparing the results obtained with these presented in [5], [6], [7], [8], the following y_m/y_{fm} ratios were calculated:

a) for uniform connectors' distribution:

$$\frac{y_m}{y_{fm}} = 1.504\,,$$

b) for linear connectors' distribution:

$$\frac{y_m}{y_{fm}} = 1.425,$$

c) for parabolic connectors' distribution:

$$\frac{y_m}{y_{fm}} = 1.399,$$

where:

y_m – the displacement at the midspan of partially composite beam,
y_{fm} – the displacement at the midspan of fully composite beam.

4. GENERAL CONCLUSIONS

1. In this article, a way of calculation of partially composite beams is presented. This way is based on numerical solution of theoretically derived systems of differential equations for simply supported composite girders.
2. The presented way of arriving at this solution may be used for each static scheme and for each composite girder (with each connection feature).
3. Differential equations for three different cases of partial connection are presented. The largest bending stiffness was achieved at parabolic connectors' distribution.
4. The bending stiffness of the beam may be increased when the connectors are concentrated towards supports.
5. The authors decided to carry out a numerical analysis, even though there was a possibility of finding an exact solution for two of three cases presented. This way of analysis allow them to find further solutions for more complicated problems with fewer number of simplifying assumptions. This way of analysis requires writing a special computer program.

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