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MACROSCOPIC PROPERTIES OF AN ORTHOTROPIC ELASTIC MEDIUM CONTAINING ARBITRARILY ORIENTED CRACKS: APPLICATION TO DAMAGE

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Abstract: An analytical method for the determination of the Eshelby tensor $\mathbf{S}$ associated with an arbitrarily oriented crack in an orthotropic elastic medium is first presented. The crack is modelled as an infinite cylinder (with low aspect ratio) along one symmetry axis of the solid matrix. The proposed methodology yields explicit expressions of the Eshelby tensor (or equivalently the Hill tensor $\mathbf{P}$) which show the interaction between the structural anisotropy and the cracks-induced anisotropy. These results are then introduced into a Mori–Tanaka homogenization scheme in order to determine the macroscopic quantities (stiffness tensor, energy) of the cracked media. A last part of the study is devoted to the formulation of an anisotropic damage model based on the micromechanical results. The ability of this model to describe the inelastic behaviour of brittle matrix composites is demonstrated. Moreover, quantitative comparisons with experimental data on a Ceramic Matrix Composite (unidirectional SiC–SiC) are provided.

Streszczenie: Przedstawiono analityczną metodę wyznaczenia tensora $\mathbf{S}$ Eshelbyego sprzężonego z dowolnie zorientowanym pęknięciem w ortotropowym ośrodku sprężystym. Pęknięcie to jest modelowane jako nieskończony walec (z małym wydłużeniem) wzdłuż jednej osi symetrii sztywnego wzmocnienia. Dzięki zaproponowanej metodzie otrzymano jawne wyrażenia tensora Eshelbyego (lub równoważnie tensor $\mathbf{P}$ Hilla), które pokazują interakcję między strukturalną anizotropią a anizotropią spowodowaną przez pęknięcia. Wyniki te wprowadzono następnie do układu homogenizacyjnego Moriego–Tanaki, aby wyznaczyć makroskopowe wielkości (tensor sztywności, energię) spękanego ośrodka. Ostatnią część artykułu poświęcono stworzeniu modelu anizotropowych uszkodzeń na podstawie danych mikromechanicznych. Wykazano, że model opisuje niesprężyste zachowanie się kompozytów z kruchym wzmocnieniem. Porównano także ilościowe i doświadczalne dane dotyczące kompozytu z ceramicznym wzmocnieniem (jednokierunkowy SiC–SiC).

Резюме: Представлен аналитический метод определения тензора $\mathbf{S}$ Эшельби, сопряженного с произвольно ориентированной трещиной в ортотропной упругой среде. Эта трещина была моделирована как бесконечный цилиндр (с малым удлинением) вдоль одной оси симметрии жесткого укрепления. Благодаря предполагаемому методу были получены явные выражения тенсора Эшельби (или тенсор $\mathbf{P}$ Гилла), которые показывают взаимодействие между структурной анизотропией, вызванной трещинами. Затем эти результаты были введены в гомогенную систему Мори–Танаки для определения микроскопических размеров (тенсора жесткости, энергии) трещиноватой среды. Последняя часть настоящей статьи посвящена построению анизотропных моделей повреждений на основе микромеханических данных. Было обнаружено, что модель описывает неупругое поведение композитов с хрупким укреплением. Были также сравнены количественные и экспериментальные данные, касающиеся композита с керамическим укреплением (однонаправленный SiC–SiC).

1. INTRODUCTION

One of the basic mechanisms at the origin of inelastic deformation of a large class of materials ranging from geologic materials (sedimentary rocks) to man-made materials (concrete, ceramic matrix composites) is microcracking. Classically, the mechanical behaviour of such deteriorating materials is modelled in the framework of Continuum Damage Mechanics by using either purely macroscopic approaches or homogenization (upscaling) techniques. In the context of upscaling techniques, the determination of the overall properties of a cracked material implies the calculation of the Eshelby tensor $\mathbf{S}$ (or equivalently the Hill tensor $\mathbf{P}$) associated with any crack. Various works concern linear elastic isotropic solid matrix containing penny-shaped cracks (see, for instance, [3], [12], [13]). For the case of materials exhibiting a structural (primary) anisotropy, few results exist in literature. They all correspond to the case where the crack is in one of the symmetry planes of the solid matrix (see [7], [8], [10]). For an arbitrarily oriented crack in an anisotropic matrix, the principal difficulty comes from the fact that the Green function corresponding to anisotropic solids (from which is classically determined the analytical expression of $\mathbf{P}$) is not known in general. The objective of this study is twofold:

(i) To provide new analytical results of the Eshelby tensor Î in the case of arbitrarily oriented cracks embedded in an orthotropic solid matrix. Our approach is based on a method first introduced by KINOSHITA and MURA [6] and FAIVRE [5].

(ii) To implement the new results in an homogenization scheme, namely the Mori–Tanaka model, in order to determine the macroscopic properties of the cracked material. This scheme serves as a basis for the development of an anisotropic damage model which is then applied to a composite material. Summation convention on repeated indices is adopted.

2. HOMOGENIZATION SCHEMES APPLIED TO CRACKED MEDIA: BASIC PRINCIPLES

2.1. METHODOLOGY AND PRINCIPLES

Consider a representative elementary volume (r.e.v.) Ω composed of an orthotopic solid matrix weakened by a system of parallel microcracks whose unit normal is denoted by $\underline{n}$. Uniform strain conditions are prescribed on the boundary $\partial\Omega$ of this r.e.v.

The microscopic displacement field $\underline{\xi}$ and stress field σ are extended into the crack (fissure) which is viewed as an elastic material with the stiffness $\mathbb{C}^f = 0$ (only opened cracks are considered). The mechanical problem defined on the r.e.v. Ω therefore reads:

$$\mathrm{div}\sigma = 0\,, \tag{1a}$$

$$\sigma = \mathbb{C}(\underline{z}):\varepsilon \qquad \text{with} \qquad \begin{cases} \mathbb{C}(\underline{z}) = \mathbb{C}^f = 0 & \text{for} \quad \underline{z} \in \Omega^f\,, \\ \mathbb{C}(\underline{z}) = \mathbb{C}^s & \text{for} \quad \underline{z} \in \Omega^s\,, \end{cases} \tag{1b}$$

$$\underline{\xi} = \mathbf{E}\cdot\underline{z} \quad \text{for} \quad \underline{z} \in \partial\Omega. \tag{1c}$$

A crucial step of upscaling techniques consists in finding the localization rule which relates the microscopic strain field to the macroscopic one: $\varepsilon(\underline{x}) = \mathbb{A}(\underline{x}) : \mathbf{E}$. The macroscopic stress Σ is therefore defined by:

$$\Sigma = \overline{\sigma(\underline{x})} = \overline{\mathbb{C}(\underline{x}):\varepsilon(\underline{x})} = \overline{\mathbb{C}(\underline{x}):\mathbb{A}(\underline{x})}:\mathbf{E} = \mathbb{C}^{\mathrm{hom}}:\mathbf{E}\,. \tag{2}$$

It follows that the homogenized stiffness tensor $\mathbb{C}^{\mathrm{hom}}$ reads:

$$\mathbb{C}^{\mathrm{hom}} = \mathbb{C}^s + \varphi^f(\mathbb{C}^f - \mathbb{C}^s):\overline{\mathbb{A}} = \mathbb{C}^s:(\mathbb{I} - \varphi^f\,\overline{\mathbb{A}}^f)\,, \tag{3}$$

in which the coherence condition $\overline{\mathbb{A}(\underline{x})} = \mathbb{I}$, coming from the application of the average rule on strain, is used. φ^f is the volume fraction of cracks.

The localization problem is classically solved by taking advantage of a fundamental result of micromechanics associated with the so-called Eshelby inhomogeneity problem [4]. For an ellipsoidal inhomogeneity I embedded in a solid matrix, the microscopic strain takes the form:

$$\varepsilon^I = (\mathbb{I} + \mathbb{P}:\delta\mathbb{C})^{-1}:\mathbf{E}\,, \tag{4}$$

where $\delta\mathbb{C} = \mathbb{C}^I - \mathbb{C}^s$ ($= -\mathbb{C}^s$ for opened cracks).

For simplicity, one introduces the Eshelby tensor:

$$\mathbb{S} = \mathbb{P} : \mathbb{C}^s, \tag{5}$$

from which equation (4) is rewritten in the form:

$$\varepsilon^I = (\mathbb{I} - \mathbb{S})^{-1} : \mathbf{E}. \tag{6}$$

The first simple micromechanical model consists in making use of (6) as the localization relation; however, such a model, classically referred to as the dilute scheme, is limited to an infinitesimal concentration of inclusions. In order to overcome this limitation, a more suitable scheme is due.

2.2. THE MORI–TANAKA SCHEME APPLIED TO CRACKED MEDIA

The Mori–Tanaka approach aims at accounting for the interaction between cracks. The principle of this method consists in embedding a crack in an orthotropic solid matrix submitted, not in the macroscopic strain **E**, but in a fictitious strain $\mathbf{E}^0$. Adapting the Eshelby result, the localization relation (6) takes then the form:

$$\varepsilon^f = (\mathbf{I} - \mathbf{S})^{-1} : \mathbf{E}_0. \tag{7}$$

The application of the strain average rule gives:

$$(1 - \varphi^f)\mathbf{E}_0 + \varphi^f \varepsilon^f = \mathbf{E}. \tag{8}$$

Combination of (7) and (8) yields:

$$\mathbf{E}_0 = [(1 - \varphi^f)\mathbf{I} + \varphi^f (\mathbf{I} - \mathbf{S})^{-1}]^{-1} : \mathbf{E}. \tag{9}$$

Reporting this result in (7) yields:

$$\overline{\mathrm{A}}^f = (\mathrm{I} - \Sigma)^{-1} : [(1-\varphi^f)\mathrm{I} + \varphi^f (\mathrm{I} - \Sigma)^{-1}]^{-1}. \tag{10}$$

The Mori–Tanaka estimate of $\mathrm{Å}^{hom}$ is eventually derived from (3)

$$\mathrm{X}^{hom} = (1-\varphi^f)\mathrm{X}^s : [(1-\varphi^f)\mathrm{I} + \varphi^f (\mathrm{I} - \Sigma)^{-1}]^{-1}. \tag{11}$$

In conclusion, it appears that the determination of the homogenized stiffness (or compliance) tensor requires evaluation of the Eshelby tensor $\mathbb{S}$ (or equivalently the Hill tensor $\mathbb{P}$ given by (5)).

3. DETERMINATION OF $\mathbb{P}$ FOR AN ARBITRARILY ORIENTED CRACK IN AN ORTHOTROPIC MEDIUM

3.1. INTRODUCTION

Consider now an orthotropic solid matrix (with the stiffness tensor $\mathbb{C}^s$) weakened by a crack geometrically described in its local frame by:

$$\frac{z_1^2}{a^2} + \frac{z_2^2}{b^2} = 1, \quad -\infty < z_3 < \infty. \tag{12}$$

This corresponds to the modelling of the crack as an infinite cylinder in the direction 3, with an elliptical section and a low aspect ratio $X = b/a$ (see figure 1). Let us first recall that the components of the solid matrix stiffness tensor in the local frame (characterized by the angle θ) with respect to the symmetry axis 1 of the matrix are obtained from a classical transformation rule: $C_{ijkl} = U_{ip}U_{jq}U_{kr}U_{ls}C^{s}_{pqrs}$, where the tensor **U** reads:

$$\mathbf{U} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{13}$$

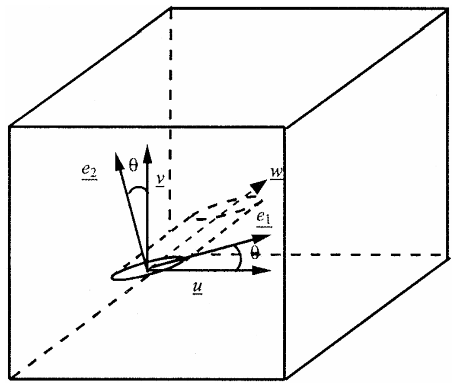

Fig. 1. Crack modelled as an infinite cylinder with elliptical section; the aspect ratio is $X = b/a$

In the local frame of the crack, the stiffness tensor then depends on the angle θ defining the crack plane and presents an apparent monoclinic symmetry. Determination of the tensor è, which is based – as previously mentioned – on the work of KINOSHITA and MURA [6], (see also FAIVRE [5], LAWS [7] or WILLIS [16]), implies that we can calculate the integral:

$$P_{ijkl} = \frac{ab}{\pi}\int_{0}^{2\pi} \frac{N_{ijkl}(\xi_1, \xi_2)}{(a^2\xi_1^2 + b^2\xi_2^2)} d\psi . \tag{14}$$

The integration is performed on the unit circle centered at the origin, in the plane (ξ, ξ_2): $|\underline{\xi}| = 1$ (e.g., $\underline{\xi} = \cos\psi\underline{e}_1 + \sin\psi\underline{e}_2$). The components of the fourth-order tensor ë read:

$$N_{ijkl}(\xi_1, \xi_2) = D_{ijkl}(\xi_1, \xi_2, 0) \tag{15}$$

with:

$$D_{ijkl} = \frac{1}{4}(\xi_i K^{-1}_{jk}\xi_l + \xi_j K^{-1}_{ik}\xi_l + \xi_i K^{-1}_{jl}\xi_k + \xi_j K^{-1}_{il}\xi_k) . \tag{16}$$

In (16), $\mathbf{K} = \underline{\xi}.\mathbb{C}.\underline{\xi}$ represents the acoustic tensor associated with $\mathbb{C}$ and the vector $\underline{\xi}$ by the relation: $K_{ik}(\underline{\xi}) = C_{ijkl}\xi_j\xi_l$. It can be noted that the anisotropy of the elastic solid matrix affects $\mathbb{P}$ through the acoustic tensor **K**. Since the above expressions imply a calculation in the local frame of the crack, $\mathbb{P}$ depends a priori on the orientation of the crack. Moreover, the components of $\mathbb{P}$ can be written in the form:

$$P_{ijkl} = \frac{1}{4}[\mathsf{M}_{ijkl} + \mathsf{M}_{jikl} + \mathsf{M}_{ijlk} + \mathsf{M}_{jilk}], \tag{17}$$

where:

$$\mathsf{M}_{ijkl} = \frac{ab}{\pi}\int_0^{2\pi} \frac{\xi_i K_{jk}^{-1}\xi_l}{(a^2\xi_1^2 + b^2\xi_2^2)}\, d\psi . \tag{18}$$

3.2. DEVELOPED METHODOLOGY FOR THE DETERMINATION OF THE TENSOR $\mathbb{P}$

The starting point here is the recent study of SUVOROV and DVORAK [14] which follows the procedure described by TING and LEE [15][1]. We consider then the two fixed unit orthogonal vectors $\underline{e}_1$ and $\underline{e}_2$ in the plane $\xi_3 = 0$; any unit vector in this plane reads then $\underline{\xi} = \cos\psi\underline{e}_1 + \sin\psi\underline{e}_2$, from which it can be verified that $\mathbf{K} = \underline{\xi}.\mathrm{C}.\underline{\xi}$ takes the form:

$$\mathbf{K} = (\cos\psi)^2\mathbf{Q} + \cos\psi\sin\psi(\mathbf{R} + \mathbf{R}^T) + (\sin\psi)^2\mathbf{T}. \tag{19}$$

Substituting $z = \cot\psi$ in this expression yields:

$$\mathbf{K}(\psi) = (\sin\psi)^2[\mathbf{Q}z^2 + z(\mathbf{R} + \mathbf{R}^T) + \mathbf{T}] = (\sin\psi)^2\mathbf{K}(z) \tag{20}$$

with:

$$\mathbf{K}(z) = z^2\mathbf{Q} + z(\mathbf{R} + \mathbf{R}^T) + \mathbf{T}. \tag{21}$$

The second-order tensors **Q**, **R**, and **T** are defined as:

$$\mathbf{Q} = \underline{e}_1.\mathbb{C}.\underline{e}_1; \quad \mathbf{R} = \underline{e}_1.\mathbb{C}.\underline{e}_2; \quad \mathbf{T} = \underline{e}_2.\mathbb{C}.\underline{e}_2. \tag{22}$$

For the calculation of $\mathbb{P}$ (equation (16)), one needs to invert $\mathbf{K}(z)$. The quantity $|\mathbf{K}(z)|$ denoting the determinant of $\mathbf{K}(z)$ and $\tilde{\mathbf{K}}(z)$ its adjoint, one has: $\mathbf{K}(z).\ \tilde{\mathbf{K}}(z) = |\mathbf{K}(z)|\ \mathbf{1}$, respectively:

[1] Note that these authors have not studied the case of arbitrarily oriented inclusions in an orthotropic solid matrix.

$$\mathbf{K}^{-1}(z) = \frac{\tilde{\mathbf{K}}(z)}{|\mathbf{K}(z)|}. \tag{23}$$

Similarly, for the tensor $\Delta = \underline{\xi} \otimes \underline{\xi}$ which with $\mathbf{K}^{-1}$ enters in definition (16), it can be verified that: $\Delta(\psi) = (\sin\psi)^2 \Delta(z)$ with $\Delta(z) = z^2 \underline{e}_1 \otimes \underline{e}_1 + z(\underline{e}_1 \otimes \underline{e}_2 + \underline{e}_2 \otimes \underline{e}_1) + \underline{e}_2 \otimes \underline{e}_2$. Moreover, we note that $a^2\xi_1^2 + b^2\xi_2^2 = (\sin\psi)^2(a^2z^2 + b^2)$.

Taking into account the variable change $z = \cot\psi$, the expression (18) reads:

$$\mathsf{M}_{ijkl} = \frac{ab}{\pi} \int_{-\infty}^{\infty} \frac{\tilde{K}_{jk}(z)\Delta_{il}(z)}{(a^2z^2 + b^2)|\mathbf{K}(z)|} dz = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{X\tilde{K}_{jk}(z)\Delta_{il}(z)}{(z^2 + X^2)|\mathbf{Q}|\, f(z)} dz, \tag{24}$$

where, with the help of (20), the identity $|\boldsymbol{K}(z)| = |\mathbf{Q}|\, f(z)$ is used. We recall that $X = \dfrac{b}{a}$ is the aspect ratio of the crack.

The interest of (24) lies in the fact that it allows the evaluation of M_{ijkl} by using the residues theorem, the function to be integrate being holomorphic out of the poles which must be determined. We note that $z = \pm iX$ are two of these poles. It is also useful to emphasize that in the general case, $f(z)$ is a polynomial function of degree 6, with three "pair" of complex conjugate roots. Obviously, these are the roots of $\mathbf{K}(z) = 0$. Noting $z_p(p = 1, 2, 3)$ the roots with positive imaginary part, one has:

$$f(z) = (z - z_1)(z - \bar{z}_1)(z - z_2)(z - \bar{z}_2)(z - z_3)(z - \bar{z}_3). \tag{25}$$

The tensor M can then be determined by the application of the residues theorem; this is done in the next section.

4. ANALYTICAL DETERMINATION OF THE TENSOR $\mathbb{P}$

4.1. EXPRESSION OF THE COMPONENTS OF HILL'S TENSOR $\mathbb{P}$

In the case of the crack with an arbitrary orientation (represented by θ) with respect to the symmetry axis 1 of the matrix, the calculation performed in the local frame of the crack leads to a polynomial function (of degree 6). This one appears as the product of 2 polynomial functions of degree 4 and degree 2:

$$|\mathbf{K}(z)| = |\mathbf{Q}|\, f(z) = f_1(z) \cdot f_2(z). \tag{26}$$

Expressions of $f_1(z)$ and $f_2(z)$ are presented in Appendix A. The determinant $|\mathbf{Q}|$ takes the form:

$$|\mathbf{Q}| = C^s_{1111} C^s_{1212} (\cos^2\theta + \alpha \sin^2\theta)(\cos^2\theta + \beta \sin^2\theta)(C^s_{3232} \sin^2\theta + C^s_{3131} \cos^2\theta). \tag{27}$$

The solutions of $f_1(z) = 0$ are:

$$\begin{aligned} z_1 &= \frac{i\sqrt{\alpha}\cos\theta - \sin\theta}{\cos\theta + i\sqrt{\alpha}\sin\theta}; \quad \bar{z}_1 = \frac{i\sqrt{\alpha}\cos\theta + \sin\theta}{-\cos\theta + i\sqrt{\alpha}\sin\theta}; \\ z_2 &= \frac{i\sqrt{\beta}\cos\theta - \sin\theta}{\cos\theta + i\sqrt{\beta}\sin\theta}; \quad \bar{z}_2 = \frac{i\sqrt{\beta}\cos\theta + \sin\theta}{-\cos\theta + i\sqrt{\beta}\sin\theta} \end{aligned} \tag{28}$$

with α and β, the complex conjugate roots of the characteristic equation of the orthotropic 2D solid:

$$C^s_{1111}C^s_{1212} - (C^s_{1111}C^s_{2222} - C^{s\;\;2}_{1122} - 2C^s_{1122}C^s_{1212})x + C^s_{2222}C^s_{1212} = 0. \tag{29}$$

The roots of $f_2 = 0$ read:

$$\begin{aligned} z_3 &= -\frac{\sin(2\theta)(C^s_{3232} + C^s_{3131}) + 2i\sqrt{C^s_{3232}C^s_{3131}}\cos(2\theta)}{2(C^s_{3131}\cos^2\theta + C^s_{3232}\sin^2\theta)}; \\ \bar{z}_3 &= -\frac{\sin(2\theta)(C^s_{3232} + C^s_{3131}) - 2i\sqrt{C^s_{3232}C^s_{3131}}\cos(2\theta)}{2(C^s_{3131}\cos^2\theta + C^s_{3232}\sin^2\theta)}. \end{aligned} \tag{30}$$

Then, by identification with (28) and (30), $|\mathbf{K}(z)|$ can be rewritten in the form:

$$|\mathbf{K}(z)| = |\mathbf{Q}|(z - z_1)(z - \bar{z}_1)(z - z_2)(z - \bar{z}_2)(z - z_3)(z - \bar{z}_3). \tag{31}$$

The poles being assumed distinct, the expression of M (equation (24)) reads:

$$\mathsf{M}_{ijkl} = 2i\left\{\frac{1}{2i}\frac{\tilde{K}_{jk}(iX)\Delta_{il}(iX)}{|\mathbf{Q}|\,f(iX)} + \sum_{i=1}^{3}\frac{\tilde{K}_{jk}(z_i)\Delta_{il}(z_i)X}{|\mathbf{Q}|\,f'(z_i)(X^2 + z_i^2)}\right\}, \tag{32}$$

for which it is recalled that $\tilde{K}_{jk}$ are the components of the adjoint of $\mathbf{K}$.

The approximation at the first order in X at the value $X = 0$ gives for (32):

$$\begin{aligned} \mathsf{M}_{ijkl} &= \frac{\tilde{K}_{jk}(0)\Delta_{il}(0)}{f(0)|\mathbf{Q}|} \\ &+ \frac{2iX}{|\mathbf{Q}|}\left\{\sum_{i=1}^{3}\frac{\tilde{K}_{jk}(z_i)\Delta_{il}(z_i)}{f'(z_i)z_i^2} + \frac{1}{2f(0)}\left[\tilde{K}'_{jk}(0)\Delta_{il}(0) + \tilde{K}_{jk}(0)\Delta'_{il}(0) - \tilde{K}_{jk}(0)\Delta_{il}(0)\frac{f'(0)}{f(0)}\right]\right\}. \end{aligned} \tag{33}$$

The imaginary part of M is null:

$$\Im(\mathsf{M}_{ijkl}) = \frac{2X}{|\mathbf{Q}|}\left\{\Re\left[\frac{\tilde{K}_{jk}(z_1)\Delta_{il}(z_1)}{f'(z_1)z_1^2}\right] + \Re\left[\frac{\tilde{K}_{jk}(z_2)\Delta_{il}(z_2)}{f'(z_2)z_2^2}\right] + \Re\left[\frac{\tilde{K}_{jk}(z_3)\Delta_{il}(z_3)}{f'(z_3)z_3^2}\right]\right\}$$
$$+ \frac{X}{2f(0)|\mathbf{Q}|}\left\{\left[\tilde{K}'_{jk}(0)\Delta_{il}(0) + \tilde{K}_{jk}(0)\Delta'_{il}(0) - \tilde{K}_{jk}(0)\Delta_{il}(0)\frac{f'(0)}{f(0)}\right]\right\} = 0 \quad (34)$$

and its real part gives :

$$\mathsf{M}_{ijkl} = \Re(\mathsf{M}_{ijkl}) = \frac{\tilde{K}_{jk}(0)\Delta_{il}(0)}{f(0)|\mathbf{Q}|} - \frac{2X}{|\mathbf{Q}|}\Im\sum_{i=1}^{3}\frac{\tilde{K}_{jk}(z_i)\Delta_{il}(z_i)}{f'(z_i)z_i^2}. \quad (35)$$

Finally, these results being established in the local frame of the crack, a last change of basis allows us to express $\mathbf{P}$ in the global frame defined by the symmetry axes of the solid matrix. The detailed expressions of the nine components of $\mathbf{P}$ in this global frame are given in Appendix A (equations (A.4)).

4.2. VALIDATION OF THE RESULTS OBTAINED

The first validation of these results is done by considering a system of parallel cracks whose orientation coincides with the symmetry axis 1 of the solid matrix. This case has been studied by LAWS [7], who obtained the following non-zero components for the tensor $\mathbf{P}$:

$$P_{1111} = \frac{C^s_{2222} + C^s_{1212}\sqrt{\alpha\beta}}{C^s_{1111}C^s_{1212}\sqrt{\alpha\beta}(\sqrt{\alpha}+\sqrt{\beta})}X; \quad P_{2222} = \frac{1}{C^s_{2222}} + \frac{C^s_{2222} - C^s_{1212}(\alpha+\beta+\sqrt{\alpha\beta})}{C^s_{2222}C^s_{1212}\sqrt{\alpha\beta}(\sqrt{\alpha}+\sqrt{\beta})}X;$$

$$P_{1122} = P_{2211} = -\frac{C^s_{1122} + C^s_{1212}}{C^s_{1111}C^s_{1212}\sqrt{\alpha\beta}(\sqrt{\alpha}+\sqrt{\beta})}X; \quad P_{3232} = \frac{1}{4C^s_{3232}} - \frac{\sqrt{C^s_{3131}}}{4{C^s_{3232}}^{3/2}}X; \quad (36)$$

$$P_{3131} = \frac{1}{4\sqrt{C^s_{3232}C^s_{3131}}}X; \quad P_{1212} = \frac{1}{4C^s_{1212}} - \frac{C^s_{1111}C^s_{2222} - {C^s_{1122}}^2}{2C^s_{1111}{C^s_{1212}}^2\sqrt{\alpha\beta}(\sqrt{\alpha}+\sqrt{\beta})}X.$$

It can be easily verified that these expressions appear as a particular case (θ = 0) of the general expressions (A.4) which were established in the present study.

In order to determine the macroscopic properties of the orthotropic solid matrix weakened by cracks, such general expressions are now incorporated in the Mori–Tanaka scheme already presented in subsection 2.2.

4.3. THE MORI–TANAKA ESTIMATE OF THE HOMOGENIZED COMPLIANCE

The starting point is equation (11) which gives the Mori–Tanaka estimate of the the homogenized stiffness tensor of the cracked medium. As in [2], let us first introduce the 2D cracks density parameter $d = \mathrm{N\,a}^2$ where N denotes the crack density (the number of cracks per unit area) of the considered set of parallel cracks. The cracks volume concentration reads then $\varphi^f = \pi N ab = \pi dX$. Noting that $\varphi^f \ll 1$ and taking the limit for low aspect ratio ($\lim_{X\to 0}$), the homogenized stiffness tensor (see, for instance, [3] for the case of isotropic solid matrix) given by (11) takes the form:

$$\mathbf{C}^{\mathrm{hom}} = \mathbf{C}^s : (\mathbf{I} + \pi d\, \mathbf{T})^{-1}, \quad \text{where} \quad \mathbf{T} = \lim_{X\to 0} X[\mathbf{I} - \mathbf{S}(X, \underline{n})]^{-1}, \tag{37}$$

in which $\underline{n}$ is the unit normal to the considered parallel cracks system.

By inversion, it follows that the compliance homogenized tensor can be put in the form:

$$\mathbf{S}^{\mathrm{hom}} = (\mathbf{I} + \pi\, d\, \mathbf{T}) : (\mathbf{S}^s) \quad \text{with} \quad \mathbf{T} = \lim_{X\to 0} X[\mathbf{I} - \mathbf{S}(X, \underline{n})]^{-1}, \tag{38}$$

in which $\mathbf{S}^s = (\mathbf{C}^s)^{-1}$ represents the compliance tensor of the solid matrix. For a parallel cracks system oriented with an angle θ (see figure 1), the non-zero components of the homogenized compliance tensor $\mathbf{S}^{\mathrm{hom}}$ are:

$$S_{1111}^{\mathrm{hom}} = S_{1111}^s + d\pi \frac{(S_{2222}^s S_{3333}^s - S_{2233}^s{}^2)(\sqrt{\alpha}+\sqrt{\beta})\sqrt{\alpha\beta}(\sin\theta)^2}{S_{3333}^s};$$

$$S_{1122}^{\mathrm{hom}} = S_{2211}^{\mathrm{hom}} = S_{1122}^s; \quad S_{1133}^{\mathrm{hom}} = S_{3311}^{\mathrm{hom}} = S_{1133}^s; \quad S_{2233}^{\mathrm{hom}} = S_{3322}^{\mathrm{hom}} = S_{2233}^s; \quad S_{3333}^{\mathrm{hom}} = S_{3333}^s;$$

$$S_{2222}^{\mathrm{hom}} = S_{2222}^s + d\pi \frac{(S_{2222}^s S_{3333}^s - S_{2233}^s{}^2)(\sqrt{\alpha}+\sqrt{\beta})(\cos\theta)^2}{S_{3333}^s};$$

$$S_{1112}^{\mathrm{hom}} = S_{2111}^{\mathrm{hom}} = S_{1112}^s + d\pi \frac{(S_{2233}^s{}^2 - S_{2222}^s S_{3333}^s)(\sqrt{\alpha}+\sqrt{\beta})\sqrt{\alpha\beta}\sin\theta\cos\theta}{2S_{3333}^s}; \tag{39}$$

$$S_{2212}^{\mathrm{hom}} = S_{1222}^{\mathrm{hom}} = S_{2212}^s + d\pi \frac{(S_{2233}^s{}^2 - S_{2222}^s S_{3333}^s)(\sqrt{\alpha}+\sqrt{\beta})\sin\theta\cos\theta}{2S_{3333}^s};$$

$$S_{3232}^{\mathrm{hom}} = S_{3232}^s + d\pi(\sin\theta)^2 \frac{\sqrt{S_{3232}^s S_{3131}^s}}{2}; \quad S_{3131}^{\mathrm{hom}} = S_{3131}^s + d\pi(\cos\theta)^2 \frac{\sqrt{S_{3232}^s S_{3131}^s}}{2};$$

$$S_{3231}^{\mathrm{hom}} = S_{3132}^{\mathrm{hom}} = S_{3231}^s + d\pi\sin\theta\cos\theta \frac{\sqrt{S_{3232}^s S_{3131}^s}}{2};$$

$$S_{1212}^{\mathrm{hom}} = S_{1212}^{s} + d\pi \frac{({S_{2233}^{s}}^{2} - S_{2222}^{s}S_{3333}^{s})(\sqrt{\alpha}+\sqrt{\beta})[(\sin\theta)^{2} + \sqrt{\alpha\beta}(\cos\theta)^{2}]}{S_{3333}^{s}}. \quad (39)$$

Note that the effect of cracks on the components of the homogenized compliance tensor depends on the elastic properties of the solid matrix and greatly on the crack orientation (defined by θ). Such a dependence accounts for the interaction between the initial anisotropy and crack's orientation.

5. A 2D MICROMECHANICAL DAMAGE MODEL FOR INITIALLY ORTHOTROPIC MATERIALS

In this section, we propose a micromechanical damage model for initially orthotropic materials. The model is based on the results presented in the preceding section and on the choice of a damage criterion which allows the propagation of distributed cracks to be described. Its predictive capabilities are analyzed through an application to a brittle matrix composite. Comparison with experimental data obtained by AUBARD [1] on a SiC–SiC is presented and shows the ability of the model proposed to describe the overall stress response of this composite subjected to off axis loadings.

5.1. FREE ENTHALPY. DAMAGE PROPAGATION BY CRACKS GROWTH

For simplicity we present the free enthalpy for the system of parallel cracks (denoted by *i*):

$$W^{*} = \frac{1}{2}\Sigma : \mathsf{S}^{\mathrm{hom}}(d^{i}) : \Sigma . \quad (40)$$

An estimate of $\mathbf{S}^{\mathrm{hom}}(d^{i})$ is already provided by the Mori–Tanaka scheme (see equations (38) and (39)). The first state law gives the macroscopic strain tensor $\mathbf{E}$:

$$\mathbf{E} = \frac{\partial W^{*}}{\partial \Sigma} = \mathsf{S}^{\mathrm{hom}} : \Sigma . \quad (41)$$

To complete the model one needs to adopt a damage criterion from which a damage evolution law can be derived. For this purpose, d^{i} being the damage parameter, the intrinsic dissipation reads:

$$\mathsf{D} = \frac{\partial W^{*}}{\partial d^{i}}\dot{d}^{i} = F^{d^{i}}\dot{d}^{i} \quad (42)$$

in which $F^{d^{i}}$ is the thermodynamic force (energy release rate) associated with damage:

$$F^{d^i} = \frac{\partial W^*}{\partial d^i}. \tag{43}$$

Based on the above thermodynamic arguments briefly summarized, the damage criterion for the cracks family under consideration can be put in the form:

$$f^i(F^{d^i}, d^i) = F^{d^i} - \mathrm{R}(d^i). \tag{44}$$

Here the function $\mathrm{R}(d^i)$ describes the curve of crack resistance to the damage propagation; for simplicity, the following form is chosen:

$$R(d^i) = k(1 + \eta d^i), \tag{45}$$

where k is the parameter which describes the damage threshold and η accounts for the hardening effect of the damage. The use of this criterion gives:

$$\begin{aligned} &-\text{if } F^{d^i} < \mathrm{R}(d^i), \quad \text{then } \dot{d}^i = 0 \ \text{(damage initiation and propagation)}, \\ &-\text{if } F^{d^i} = \mathrm{R}(d^i), \quad \text{then } \dot{d}^i \geq 0 \ \text{(growing damage)}. \end{aligned} \tag{46}$$

The damage evolution is obtained by assuming the normality rule:

$$\left| \begin{aligned} &\dot{d}^i = \dot{\Lambda}_{d^i} \frac{\partial f^i(F^{d^i}, d^i)}{\partial F^{d^i}} = \dot{\Lambda}_{d^i}; \quad \dot{\Lambda}_{d^i} \geq 0, \\ &\text{so } \dot{d}^i = \begin{cases} 0 & \text{if } f^i < 0, \quad \text{where } (f^i = 0 \text{ and } \dot{f}^i < 0), \\ \dot{\Lambda}_{d^i} & \text{if } f^i = 0 \quad \text{and } \dot{f}^i = 0. \end{cases} \end{aligned} \right. \tag{47}$$

The damage multiplier $\dot{\Lambda}_{d^i}$ is derived from the classical Kuhn–Tucker consistence condition: $\dot{f} = 0$.

5.2. APPLICATION OF THE MODEL TO AN UNIDIRECTIONAL SiC–SiC COMPOSITE

The purpose now is to apply the model being proposed to study the response of a brittle matrix composite subjected to an uniaxial tensile loading, with the objective to check the predictive capabilities of the model. The experimental data come from AUBARD [1]; this experimental study concerns an unidirectional SiC–SiC composite subjected to an off-axis tensile loading in different directions (described by the angle φ) with respect to one of the symmetry axis of the material (see figure 2). The elastic moduli of the solid matrix, which is assumed to correspond to the initial elastic moduli of the composite, are: E_1 = 320000 MPa, E_2 = 170000 MPa for the Young modulus, G_{12} = 90000 MPa for the shear modulus and ν_{12} = 0.18 for the Poisson ratio.

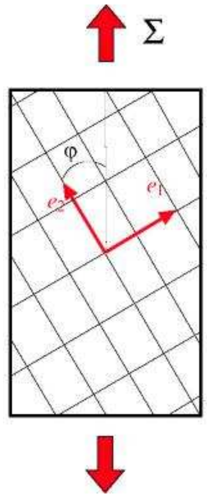

Fig. 2. Representation of tensile test applied with the angle φ

The application of damage model implies also identification of the parameters k and η involved in the damage criterion, and of an initial value for the crack density parameter. The identification procedure followed here consists in calibrating the model parameters on the angle $\varphi = 0°$ tensile test. The validation of the model is done from simulations performed on off-axis tests (with two angles of 20° and 45°). Besides, the two model parameters entering in the damage criterion have the values k = 3.75 J.m^{-2} and η = 140, and the initial crack density parameter d_0 = 0.01. Finally, it must be emphasized that the simulations were performed with a number of 60 distinct crack families uniformly distributed.

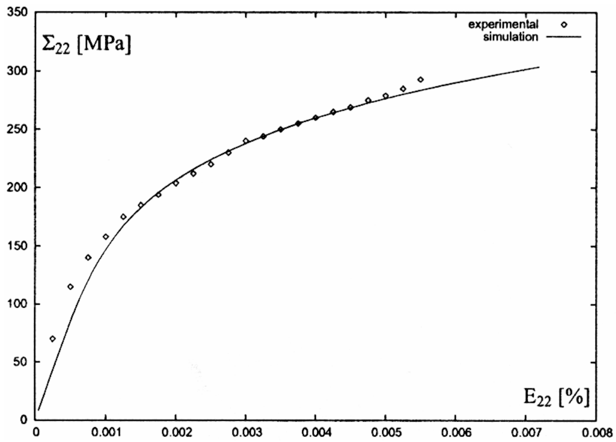

Fig. 3. Tensile test of SiC–SiC composite with $\theta = 0°$

Figure 3 shows the stress–strain response for the test with $\varphi = 0°$; the good agreement with the experimental data proves only the coherence of the identification procedure. The results predicted for the off-axis loading tests are presented in figure 4. A good agreement with experimental data is again observed. In particular, the relative positions of the three curves (shown in figure 5) appear as the consequence of interaction between damage-induced and initial anisotropy.

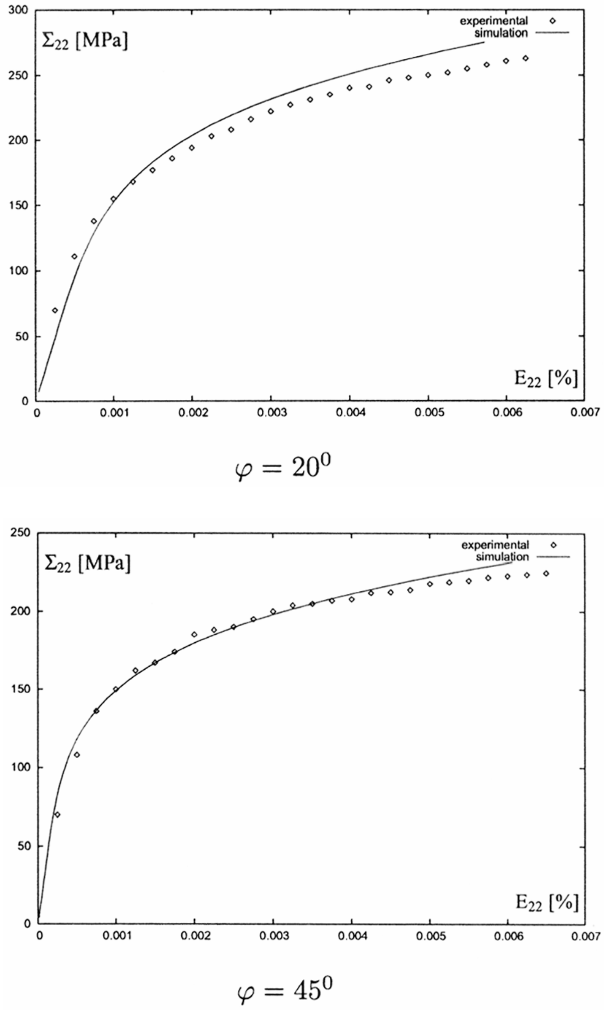

Fig. 4. Off-axis traction tests of unidirectional SiC–SiC composite

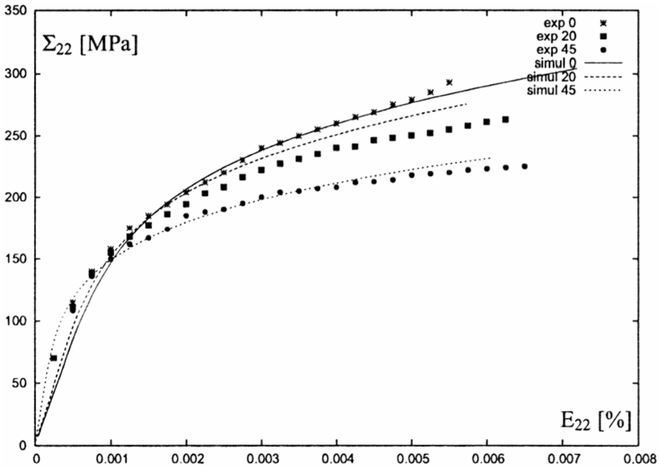

Fig. 5. Comparison of the tensile tests of unidirectional SiC–SiC composite

6. CONCLUSIONS

The present study is devoted to a micromechanical analysis of an orthotropic medium containing arbitrarily oriented cracks. The procedure proposed has allowed us to establish general explicit expressions of the Eshelby (or Hill) tensor associated with these cracks. The analytical results clearly show the interaction between the cracks orientation and the initial anisotropy of the material. A consequence of this interaction is the loss of material symmetry due to the presence of cracks. On the basis of the results obtained, a new damage model for orthotropic materials is proposed, using a Mori–Tanaka homogenization scheme and a damage criterion based on energy release rate. Numerical predictions of the damage model compare well with experimental data on a Ceramic Matrix Composite (SiC–SiC). Current works on this model deal with a description of cracks closure process which classically occurs when the composite is submitted to a compression loading. Finally, introduction of friction phenomena on closed cracks will be the focus of future developments.

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APPENDIX A. ANALYTICAL EXPRESSIONS OF $|\mathbf{K}(z)|$ AND THE COMPONENTS OF THE TENSOR $\mathbb{P}$

A.1. EXPRESSIONS OF $f_1(z)$, $f_2(z)$ IN $|\mathbf{K}(z)|$

The two polynomial functions $f_1(z)$, $f_2(z)$ are respectively:

$$\begin{aligned} f_1(z) = {} & [C^s_{1111}C^s_{1212}(\cos\theta)^4 + C^s_{2222}C^s_{1212}(\sin\theta)^4 + \chi(\cos\theta)^2(\sin\theta)^2]z^4 \\ & +\{[2C^s_{1111}C^s_{1212}(\cos\theta)^2 - 2C^s_{2222}C^s_{1212}(\sin\theta)^2] - \chi\cos 2\theta\}\sin 2\theta z^3 \\ & +\{\chi[1-6(\cos\theta)^2(\sin\theta)^2] + 6C^s_{1212}(C^s_{1111}+C^s_{2222})(\cos\theta)^2(\sin\theta)^2\}z^2 \\ & +[2C^s_{1111}C^s_{1212}(\sin\theta)^2 + 2C^s_{2222}C^s_{1212}(\cos\theta)^2 + \chi\cos 2\theta]\sin 2\theta z \\ & + C^s_{1111}C^s_{1212}(\sin\theta)^4 + C^s_{2222}C^s_{1212}(\cos\theta)^4 + \chi(\sin\theta)^2(\cos\theta)^2, \end{aligned} \tag{A.1}$$

where:

$$\chi = (C^s_{1111}C^s_{2222} - {C^s_{1122}}^2 - 2C^s_{1122}C^s_{1212}),$$

and:

$$f_2(z) = (C^s_{3232}\sin^2\theta + C^s_{3131}\cos^2\theta)z^2 + (C^s_{3232} + C^s_{3131})\sin 2\theta z + C^s_{3232}\cos^2\theta + C^s_{3131}\sin^2\theta. \quad \text{(A.2)}$$

Note that, in order to make easier the search of the roots of $f_1(z) = 0$, we introduced the variable change $z = (u\cos\theta - \sin\theta)/(\cos\theta + u\sin\theta)$, which allowed us to write (A.1) as:

$$f_1(z) = \frac{C^s_{1111}C^s_{1212}u^4 + (C^s_{1111}C^s_{2222} - C^{s\,2}_{1122} - 2C^s_{1122}C^s_{1212})u^2 + C^s_{2222}C^s_{1212}}{(\cos\theta + u\sin\theta)^4}. \quad \text{(A.3)}$$

A.2. COMPONENTS OF $\mathbf{P}$

The determination of the components P_{ijkl} is performed by combination of (17) and (35). In the frame defined by the material symmetry axes, the following expressions are obtained:

$$P_{1111} = \frac{[C^s_{2222}(\cos\theta)^2 + C^s_{1212}(\sin\theta)^2](\sin\theta)^2}{C^s_{1111}C^s_{1212}[(\sin\theta)^4 + \alpha\beta(\cos\theta)^4 + (\alpha+\beta)(\cos\theta)^2(\sin\theta)^2]} + \frac{X}{C^s_{1111}C^s_{1212}(\alpha-\beta)} \cdot \left\{\frac{\sqrt{\alpha}[\alpha(\cos\theta)^2 - (\sin\theta)^2](\alpha C^s_{1212} - C^s_{2222})}{[\alpha(\cos\theta)^2 + (\sin\theta)^2]^2} - \frac{\sqrt{\beta}[\beta(\cos\theta)^2 - (\sin\theta)^2](\beta C^s_{1212} - C^s_{2222})}{[\beta(\cos\theta)^2 + (\sin\theta)^2]^2}\right\};$$

$$P_{1122} = \frac{(C^s_{1122} + C^s_{1212})(\cos\theta)^2(\sin\theta)^2}{C^s_{1111}C^s_{1212}[(\sin\theta)^4 + \alpha\beta(\cos\theta)^4 + (\alpha+\beta)(\cos\theta)^2(\sin\theta)^2]} + \frac{X(C^s_{1122} + C^s_{1212}}{C^s_{1111}C^s_{1212}(\alpha-\beta)} \cdot \left\{\frac{\sqrt{\alpha}[\alpha(\cos\theta)^2 - (\sin\theta)^2]}{[\alpha(\cos\theta)^2 + (\sin\theta)^2]^2} - \frac{\sqrt{\beta}[\beta(\cos\theta)^2 - (\sin\theta)^2]}{[\beta(\cos\theta)^2 + (\sin\theta)^2]^2}\right\};$$

$$P_{2222} = \frac{[C^s_{1111}(\sin\theta)^2 + C^s_{1212}(\cos\theta)^2](\cos\theta)^2}{C^s_{1111}C^s_{1212}[(\sin\theta)^4 + \alpha\beta(\cos\theta)^4 + (\alpha+\beta)(\cos\theta)^2(\sin\theta)^2]} + \frac{X}{C^s_{1111}C^s_{1212}(\alpha-\beta)} \cdot \left\{\frac{[\alpha(\cos\theta)^2 - (\sin\theta)^2](C^s_{1212} - C^s_{1111})}{\sqrt{\alpha}[\alpha(\cos\theta)^2 + (\sin\theta)^2]^2} - \frac{[\alpha(\cos\theta)^2 - (\sin\theta)^2](C^s_{1212} - \beta C^s_{2222})}{\sqrt{\beta}[\beta(\cos\theta)^2 + (\sin\theta)^2]^2}\right\}; \quad \text{(A.4)}$$

$$P_{1112} = \frac{[C^s_{2222}(\cos\theta)^2 - C^s_{1122}(\sin\theta)^2]\sin 2\theta}{4C^s_{1111}C^s_{1212}[(\sin\theta)^4 + \alpha\beta(\cos\theta)^4 + (\alpha+\beta)(\cos\theta)^2(\sin\theta)^2]} + \frac{X\sin 2\theta}{2C^s_{1111}C^s_{1212}(\alpha-\beta)} \cdot \left\{\frac{\sqrt{\alpha}(\alpha C^s_{1122} + C^s_{2222})]}{[\alpha(\cos\theta)^2 + \sin^2\theta]^2} - \frac{\sqrt{\beta}(\beta C^s_{1122} + C^s_{2222})}{[\beta(\cos\theta)^2 + (\sin\theta)^2]^2}\right\};$$

$$P_{2212} = \frac{[C^s_{1111}(\sin\theta)^2 - C^s_{1122}(\cos\theta)^2]\sin 2\theta}{4C^s_{1111}C^s_{1212}[(\sin\theta)^4 + \alpha\beta(\cos\theta)^4 + (\alpha+\beta)(\cos\theta)^2(\sin\theta)^2]} + \frac{X\sin 2\theta}{2C^s_{1111}C^s_{1212}(\alpha-\beta)} \cdot \left\{\frac{\sqrt{\alpha}(\alpha C^s_{1111} + C^s_{1122})}{[\alpha(\cos\theta)^2 + (\sin\theta)^2]^2} - \frac{\sqrt{\beta}(\beta C^s_{1111} + C^s_{1122})}{[\beta(\cos\theta)^2 + (\sin\theta)^2]^2}\right\};$$

$$P_{1212} = \frac{[C^s_{1111}(\sin\theta)^4 + C^s_{2222}(\cos\theta)^4 - C^s_{1122}(\sin\theta)^2(\cos\theta)^2]}{4C^s_{1111}C^s_{1212}[(\sin\theta)^4 + \alpha\beta(\cos\theta)^4 + (\alpha+\beta)(\cos\theta)^2(\sin\theta)^2]} + \frac{X(C^s_{1111}C^s_{2222} - {C^s_{1122}}^2)}{4C^s_{1111}{C^s_{1212}}^2(\alpha-\beta)}$$
$$\cdot\left\{\frac{\sqrt{\alpha}[\alpha(\cos\theta)^2 - (\sin\theta)^2]}{[\alpha(\cos\theta)^2 + (\sin\theta)^2]^2} - \frac{\sqrt{\beta}[\beta(\cos\theta)^2]}{[\beta(\cos\theta)^2 + (\sin\theta)^2]^2}\right\};$$

$$P_{1313} = \frac{(\sin\theta)^2}{4[C^s_{3232}(\cos\theta)^2 + C^s_{3131}(\sin\theta)^2]} + \frac{X}{4}\sqrt{\frac{C^s_{3232}}{C^s_{3131}}}\,\frac{[C^s_{3232}(\cos\theta)^2 - C^s_{3131}(\sin\theta)^2]}{[C^s_{3232}(\cos\theta)^2 + C^s_{3131}(\sin\theta)^2]^2};$$

$$P_{2323} = \frac{(\cos\theta)^2}{4[C^s_{3232}(\cos\theta)^2 + C^s_{3131}(\sin\theta)^2]} - \frac{X}{4}\sqrt{\frac{C^s_{3131}}{C^s_{3232}}}\,\frac{[C^s_{3232}(\cos\theta)^2 - C^s_{3131}(\sin\theta)^2]}{[C^s_{3232}(\cos\theta)^2 + C^s_{3131}(\sin\theta)^2]^2}; \tag{A.4}$$

$$P_{1323} = \frac{\sin\theta\cos\theta}{4[C^s_{3232}(\cos\theta)^2 + C^s_{3131}(\sin\theta)^2]} - \frac{X\sqrt{C^s_{3131}C^s_{3232}}\,\sin 2\theta}{4[C^s_{3232}(\cos\theta)^2 + C^s_{3131}\sin^2\theta]^2}.$$