Studia Geotechnica et Mechanica, Vol. XXIX, No. 3–4, 2007

THE SPECTRAL ANALYSIS OF A BEAM UNDER A RANDOM TRAIN OF MOVING FORCES

MAŁGORZATA GŁADYSZ, PAWEŁ ŚNIADY

Institute of Civil Engineering, Wrocław University of Technology,
50-370 Wrocław, Wybrzeże Wyspiańskiego 27, Poland.
Email: malgorzata.gladysz@pwr.wroc.pl and pawel.sniady@pwr.wroc.pl

Abstract: In this paper, the dynamic response of a beam to a random train of moving forces is considered. An analytical technique is developed to determine the spectral density function of the beam's response. The random train of moving forces forms a filtered Poisson process. In order to obtain simple algebraic relations for the spectral denisity function of the beam's response, the dynamic influence function has been introduced. As an example the spectral density functions of a bridge modelled as a simply supported beam are determined.

Streszczenie: Praca omawia zagadnienie drgań belki obciążonej losową serią ruchomych sił, tworzących proces Poissona. W rozważaniach wykorzystano analizę widmową odpowiedzi układu. Aby uzyskać proste algebraiczne związki, określające funkcję gęstości widmowej drgań belki, wprowadzono dynamiczną funkcję wpływu. W przykładzie numerycznym przedstawiono funkcje gęstości widmowej konstrukcji mostowej, której modelem jest belka swobodnie podparta.

Резюме: В настоящей работе обсужден вопрос колебаний балок, загруженных случайной серией подвижных сил, создающих процесс Пуассона. В рассуждениях использован спектральный анализ ответа системы. Для получения простых алгебраических связей, идентифицирующих функцию спектральной плотности колебаний балок введена динамичная функция влияния. В численном примере представлены функции плотности спектральной конструкции моста, которой моделью является свободно подкрепленная балка.

1. INTRODUCTION

The problem of vibration of engineering structures arising from the passage of a moving load is of great importance in dynamics of structures. Vibration of that kind, occurring mainly in bridges, has been the subject of studies for many years and there is an extensive literature, e.g., see FRYBA [1], on the subject. TUNG [2]–[4] was probably the first to publish papers on the stochastic vibrations and reliability of a bridge beam subjected to a random train of point forces. In the papers by ŚNIADY and co-authors [5]–[9], the analysis of the beam's vibrations, the estimation of the beam's reliability and fatigue modelled as the first crossing problem have been presented. The vibrations of a beam under various boundary conditions due to a train of random forces moving along the beam at a constant speed and in the same direction have been analysed by ZIBDEH and RACKWITZ [10], [11]. The problem of vibration of a suspension bridge under a random train of moving load has been discussed by BRYJA and ŚNIADY [12]–[14]. In all these aforementioned papers, the random train of moving forces has been assumed to be a Poissonian moving load process which is an analogue of the Poissonian pulse process, see also LIN [15], ROBERTS [16], [17], ŚNIADY [18], MAZUR-ŚNIADY and ŚNIADY [19], GŁADYSZ and ŚNIADY [20]. A different approach to this problem has been shown by PAOLA and RICCARDI [21], RICCARDI [22].

The loading of highway bridges is characterized by the occurrence of millions of repetitive random load events. This type of load causes material fatigue and leads to an ultimate damage of structure. Narrow-band stochastic vibration, where one frequency is dominating, is one of the factors which accelerate fatigue in bridge structures. In the problem of beam's vibrations under a random train of moving forces, we want to determine the velocity of these forces for which the response of the beam is a narrow-band stochastic process. The paper presents the spectral analysis of the beam's vibration under a random train of moving forces which forms a filtered Poisson process.

The introduction of the dynamic influence function (15) allows the authors to obtain the spectral density function of the normal mode response (19). The expressions have been used to analyse the influence of the velocity of the moving forces on the spectral density function of the simply supported beam with determinate parameters: length, mass, damping coefficient and flexural rigidity.

2. STATEMENT OF THE PROBLEM AND GENERAL SOLUTION

Let us consider vibrations of a beam of the finite length l which is a linear, elastic structure. Vibrations are caused by a train of forces moving in the same direction, all at equal, constant speed v (figure 1). The forces arrive at the beam at random times t_k which constitute a Poisson process $N(t)$ with the parameter λ. The Poisson process $N(t)$ and its increment $dN(t)$ give the number of forces arriving in the time intervals (0, t) and (t, $t + dt$), respectively.

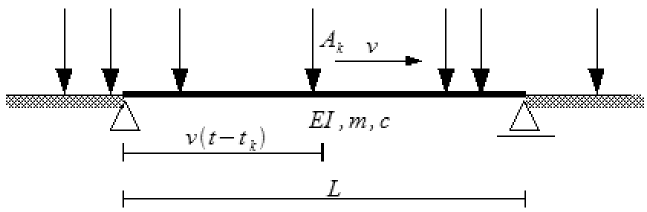

Fig. 1. Load pattern of a beam

Vibrations of the beam due to this train of forces are described by the following equation of motion:

$$EIw^{\mathrm{IV}}(x,t) + c\dot{w}(x,t) + m\ddot{w}(x,t) = \sum_{k=1}^{N(t)} A_k \delta[x - v(t - t_k)], \tag{1}$$

where EI denotes the flexural stiffness of the beam, c and m are the damping coefficient and the mass per-unit-length of the beam, respectively, $\delta(\ .\ )$ stands for the Dirac delta function, Roman numeral superscripts denote differentiation with respect to the spatial co-ordinate and superscript dots denote differentiation with respect to time. In the loading process, the amplitudes A_k are random variables, which are mutually independent and also independent of the times t_k and their expected values $E[A_k] = E[A]$, $E[A_k^2] = E[A^2]$ are known, where the symbol $E[\bullet]$ denotes the expected value of the quantity in the brackets.

It is assumed that the deflection $w(x, t)$ of the beam is obtained as the sum of the modal components:

$$w(x,t) = \sum_{n}^{\infty} y_n(t)\, W_n(x). \tag{2}$$

The eigenfunctions satisfy the equation

$$W_n^{\mathrm{IV}}(x) - \lambda_n^4\, W_n(x) = 0 \qquad (n = 1, 2, ...) \tag{3}$$

and appropriate boundary conditions. The symbol λ_n $(n = 1, 2, ...)$ denotes the n-th eigenvalue.

After inserting expression (2) into equation (1) one obtains the set of uncoupled equations:

$$\ddot{y}_n(t) + 2\alpha\, \dot{y}_n(t) + \omega_n^2 y_n(t) = \frac{1}{m\gamma_n^2} \sum_{k=1}^{N(t)} A_k W_n[v(t - t_k)] \left[H_T(t - t_k) - H_T\left(t - t_k - \frac{l}{v}\right) \right], \tag{4}$$

where:

$$2\alpha = \frac{c}{m}, \quad \omega_n^2 = \lambda_n^4 \frac{EJ}{m}, \quad \gamma_n^2 = \int_0^l W_n^2(x)\, dx, \quad H_T(t - t_k) = \begin{cases} 1 \ \text{ for } t_k \le t \le t_k + \dfrac{l}{v} \\ 0 \ \text{ for } t < t_k,\ t > t_k + \dfrac{l}{v} \end{cases}$$

and $T = \dfrac{l}{v}$ is the time of force passage along the beam.

The steady-state solution of equation (4) can be obtained in the form of the Stieltjes stochastic integral with respect to the Poisson process $N(t)$ as [5], [15]:

$$y_n(t) = \frac{1}{m\gamma_n^2} \int_{-\infty}^{t} \int_{\tau}^{\eta} A(\tau)\, h_n(t-\xi)\, W_n[v(\xi-\tau)]\, d\xi\, dN(\tau)\,, \tag{5}$$

where: $\eta = \min\left(t, \tau + \dfrac{l}{v}\right)$, $h_n(t-\xi) = \Omega^{-1} \exp[-\alpha(t-\xi)] \sin \Omega_n (t-\xi)$ is the impulse response function and $\Omega_n^2 = \omega_n^2 - \alpha^2$ is the damped natural frequency.

After taking into account that the increments $dN(t)$ of the Poisson process satisfy relations

$$E[dN^k(t)] = \lambda\, dt \qquad \text{for } k = 1, 2, ... \tag{6}$$

and

$$E\big[dN(t_1)\, dN(t_2)\big] = \lambda^2 dt_1\, dt_2 \qquad \text{for } t_1 \neq t_2\,, \tag{7}$$

the expected value and covariance function of the normal mode $y_n(t)$ can be obtained from the expressions

$$E[y_n(t)] = \frac{E[A]\lambda}{m\gamma_n^2} \int_{-\infty}^{t} \int_{\tau}^{\eta} h_n(t-\xi) W_n[v(\xi-\tau)]\, d\xi\, d\tau \tag{8}$$

and

$$C_{y_n y_n}(t_1, t_2) = \frac{E[A^2]\lambda}{m^2\gamma_n^4} \int_{-\infty}^{t} \int_{\tau}^{\eta_1} \int_{\tau}^{\eta_2} h_n(t_1-\xi_1) h_n(t_2-\xi_2)$$
$$\cdot W_n[v(\xi_1-\tau)]\, W_n[v(\xi_2-\tau)]\, d\xi_1 d\xi_2 d\tau, \tag{9}$$

where $t = \min(t_1, t_2)$.

Now, our aim is to find the expression for the spectral density function $\Phi_{y_n y_n}(\omega)$ using the covariance function given by equation (9). In order to present the method of solving the problem, we shall consider the stationary vibrations of a system with one degree of freedom described by an equation similar to (4), namely

$$\ddot{y}(t) + 2\alpha\, \dot{y}(t) + \omega_0^2 y(t) = \frac{1}{m} X(t)\,, \tag{10}$$

where the excitation process $X(t)$ is a weakly stationary stochastic process.

For a weakly stationary stochastic processes $X(t)$ the relationships between the covariance function $C_{XX}(t)$ and the spectral density function $\Phi_{XX}(\omega)$ have well-known forms [15], [23]:

$$C_{XX}(t) = \int_{-\infty}^{\infty} \Phi_{XX}(\omega)\, e^{i\omega t}\, d\omega \tag{11}$$

and

$$\Phi_{XX}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C_{XX}(t)\, e^{-i\omega t}\, dt\,, \tag{12}$$

where $i = \sqrt{-1}$.

The equation

$$\Phi_{YY}(\omega) = H(\omega)\Phi_{XX}(\omega)H^{*}(\omega) \tag{13}$$

relates the input $X(t)$ and the output $y(t)$ spectral density functions through an algebraic equation (see [15], [23]). The function $H(\omega)$ is the frequency response function and has the form

$$H(\omega) = \frac{1}{m(\omega_0^2 - \omega^2 + 2i\alpha\omega)}\,, \tag{14}$$

and the superscript * indicates the complex conjugate.

Notice that in the case of equation (4) it is not possible to find the spectral density function in a simple algebraic form analogous to (13) because the response of the system is a filtered Poisson process and the covariance function (9) has a more complex form than when the dynamic system is excitated by a weakly stationary stochastic process. To overcome this difficulty let us introduce the dynamic influence function $G_n(t)$ which is the beam's normal mode response at the time t to the force $A_k = 1$ moving at the velocity v. The function $G_n(t)$ can be obtained from the integral:

$$G_n(t-\tau) = \begin{cases} G_n^{(1)}(t-\tau) = \dfrac{1}{m\gamma_n^2} \displaystyle\int_{\tau}^{t} h_n(t-\xi) W_n[v(\xi-\tau)]\, d\xi & \text{for } t - \dfrac{l}{v} \le \tau \le t, \\[2ex] G_n^{(2)}(t-\tau) = \dfrac{1}{m\gamma_n^2} \displaystyle\int_{\tau}^{\tau+\frac{l}{v}} h_n(t-\xi) W_n[v(\xi-\tau)]\, d\xi & \text{for } 0 \le \tau \le t - \dfrac{l}{v} \text{ and } t \ge \dfrac{l}{v}. \end{cases} \tag{15}$$

The function $G_n^{(1)}(t-\tau)$ is the normal mode response at the time t to the moving force $A_k = 1$, the arrival time is

$$\tau \in \left(t - \frac{l}{v},\, t \right)$$

and the force is acting on the beam (force vibration), whereas the function $G_n^{(2)}(t-\tau)$ is the normal mode response to the force that left the beam, i.e.,

$$\tau \in \left(0,\, t-\frac{l}{v}\right)$$

(free vibration). Now, using the dynamic influence function, the covariance function for steady-state normal mode response has the form:

$$\begin{aligned} C_{y_n y_n}(t_1, t_2) &= E[A^2]\,\lambda \int_{-\infty}^{t} G_n(t_1-\tau)\, G_n(t_2-\tau)\, d\tau \\ &= E[A^2]\,\lambda \int_{0}^{\infty} G_n(\xi)\, G_n(t_2-t_1+\xi)\, d\xi \\ &= C_{y_n y_n}(t_2-t_1). \end{aligned} \tag{16}$$

As can be seen from equation (16) normal mode response $y_n(t)$ in the case under consideration is a weakly stationary stochastic process. The expression (16) can be presented in a double integral form:

$$\begin{aligned} C_{y_n y_n}(t) &= E[A^2]\,\lambda \int_{-\infty}^{t_1}\int_{-\infty}^{t_2} G_n(t_1-\xi_1)\, G_n(t_2-\xi_2)\, \delta(\xi_1-\xi_2)\, d\xi_1\, d\xi_2 \\ &= E[A^2]\,\lambda \int_{0}^{\infty}\int_{0}^{\infty} G_n(\eta_1)\, G_n(\eta_2)\, \delta(t+\eta_2-\eta_1)\, d\eta_1\, d\eta_2, \end{aligned} \tag{17}$$

where $t = t_2 - t_1$.

Relation (17) can be used to find the spectral density function $\Phi_{y_n y_n}(\omega)$ of the normal mode response $y_n(t)$. Taking into account equation (17) and the relationships between the covariance function and the spectral density function ((11) and (12)) we obtain

$$\begin{aligned} \Phi_{y_n y_n}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} C_{y_n y_n}(t)\, e^{-i\omega t} dt \\ &= \frac{1}{2\pi} E[A^2]\,\lambda \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} G_n(\eta_1)\, G_n(\eta_2)\, \delta(t+\eta_2-\eta_1)\, e^{-i\omega t} d\eta_1\, d\eta_2\, dt \\ &= \frac{1}{2\pi} E[A^2]\,\lambda \int_{0}^{\infty} G_n(\eta_1) e^{-i\omega\eta_1} d\eta_1 \int_{0}^{\infty} G_n(\eta_2) e^{i\omega\eta_2} d\eta_2 \int_{-\infty}^{\infty} e^{-i\omega t}\delta(t) dt. \end{aligned} \tag{18}$$

Expression (18) can be presented in a short form

$$\Phi_{y_n y_n}(\omega) = \frac{E[A^2]\lambda}{2\pi} J_n(\omega)\, J_n^*(\omega)\,, \tag{19}$$

where:

$$\begin{aligned} J_n(\omega) &= \int_0^{\infty} G_n(\eta)\, e^{-i\omega\eta}\, d\eta \\ &= \int_0^{\frac{l}{v}} G_n^{(1)}(\eta)\, e^{-i\omega\eta}\, d\eta + \int_{\frac{l}{v}}^{\infty} G_n^{(2)}(\eta) e^{-i\omega\eta}\, d\eta. \end{aligned} \tag{20}$$

The function $J_n(\omega)$ represents the Fourier transform of the dynamic influence function $G_n(t)$. Equation (19) may be regarded as an analogue of (13) for a random train of moving forces and when the response of the linear system is a filtered Poisson process.

3. PARTICURAL SOLUTION AND NUMERICAL ANALYSIS

Expressions (15) and (19) have been used to analyse the influence of the velocity of the moving forces v on the spectral density function of a simply supported beam. The analysis has been done for the first eigenfrequency of the beam; i.e., n = 1 $(\omega_n = \omega_1)$. For a simply supported beam (figure 1) the eigenvalue λ_n, the eigenfrequency and the eigenfunction are equal to:

$$\lambda_n = \frac{n\pi}{l}, \quad \omega_n = \left(\frac{n\pi}{l}\right)^2 \sqrt{\frac{EI}{m}}, \quad W_n(x) = \sin\frac{n\pi x}{l}. \tag{21}$$

It has been assumed that the parameters of the beam are as follows: the span l = 20 m, the damping ratio ξ = 0.02, where $\alpha = 0.5\, c/m = \xi\omega_1$ and the first natural frequency is in the range $\omega_1 =$ 2–10 Hz.The velocities of the moving forces are given by formulae

$$v = \frac{l}{2\pi a}\omega_1 = \frac{l}{aT_1}\,,$$

where $a > 0$ and its variability is assumed in the range of $0.2 \le a \le 2.0$.

The numerical analysis has been carried out in a real range of velocity values; i.e., the parameter a = 0.2 gives us the velocity of moving forces v = 300 km/h, for this reason we do not use the small value for the parameter α. The constant C in all figures takes the shape of

$$C = \frac{E[A^2]\lambda}{ml^2}.$$

Figures 2, 3, 4 and 5 show the graphs of the spectral density function $\Phi_{y_n y_n}(\omega)$ for selected values ω_1 and fixed value of the parameter $a = 1$. From these figures one can see that the beam response is a narrow-band process with a clear peak for $\omega = \omega_1$. The velocity of the moving forces increases together with the growth of ω_1 which allows us to estimate the value of the velocity v dangerous for the structure.

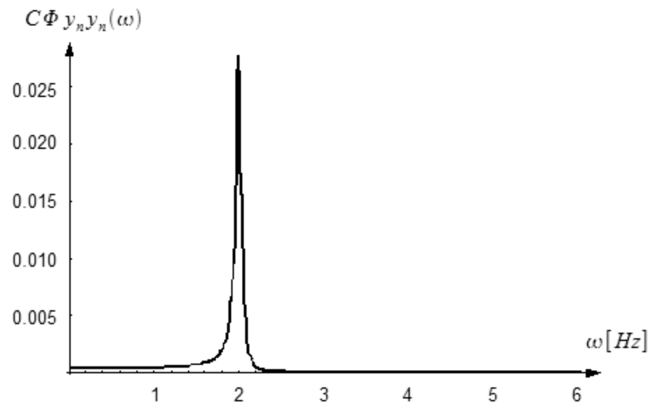

Fig. 2. The spectral density function for ω_1 = 2 Hz (v = 23 km/h)

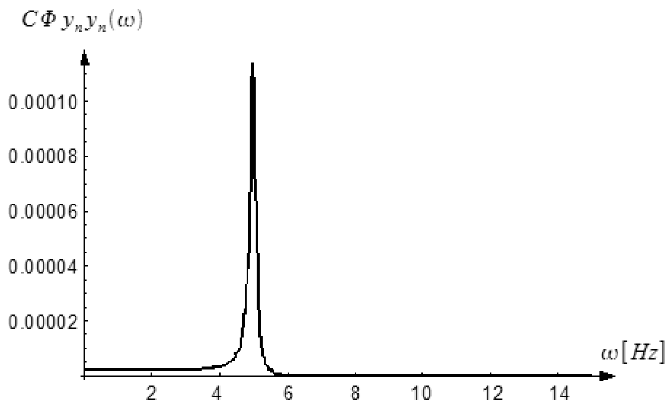

Fig. 3. The spectral density function for ω_1 = 5 Hz (v = 57 km/h)

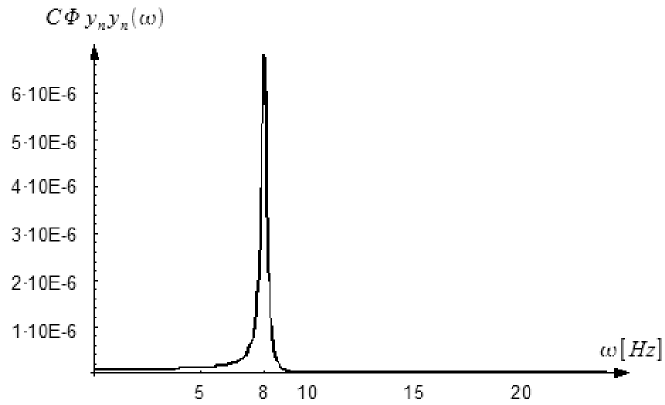

Fig. 4. The spectral density function for ω_1 = 8 Hz (v = 92 km/h)

Figures 6, 7, 8 and 9 show the graphs of the spectral density function for $\omega_1 = \omega_{\text{real}} = 5.3$ Hz. This value was calculated for the simply supported steel beam. The following assumptions were made: the length of the beam l = 20 m, the moment of inertia I is equal to 0.0007 m^4, the mass-per-unit-length m is equal to 3000 kg/m and Young's modulus E = 200 GPa (2 · 10^8 kN/m^2). Also these graphs correspond to the narrow-band process with sharp peaks for $\omega = \omega_1$. In the case of the parameter a = 2 (see figure 6), one can see the clear influence of the quasi-static action of the beam on the moving forces. Comparing the maximum values of peaks in figures 5, 6, 7 and 8 we can notice that: the values of spectral density function are tenfold smaller for a = 2 than for a = 0.5 if the time of the forces passage along the beam is equal to the half of the time of eigenvibration.

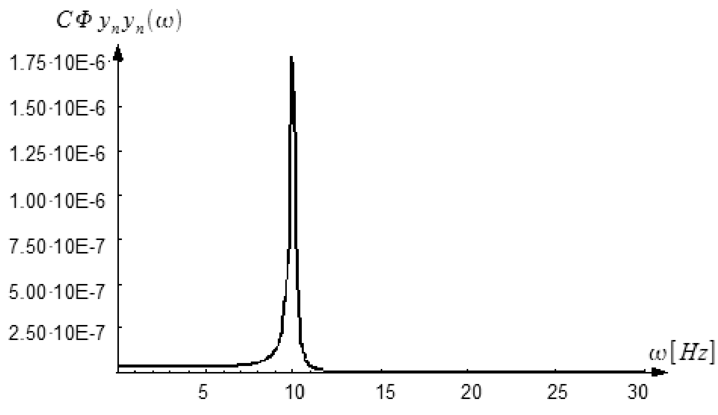

Fig. 5. The spectral density function for ω_1 = 10 Hz (v = 115 km/h)

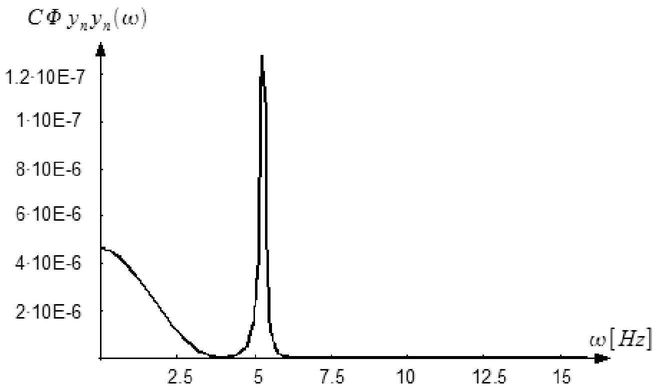

Fig. 6. The spectral density function for a = 2 (v = 30 km/h)

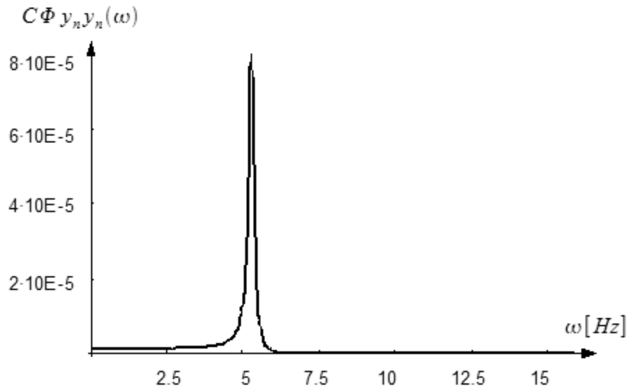

Fig. 7. The spectral density function for a = 1 (v = 61 km/h)

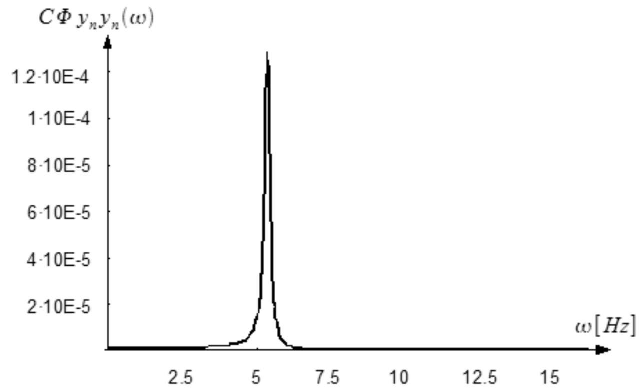

Fig. 8. The spectral density function for a = 0.75 (v = 81 km/h)

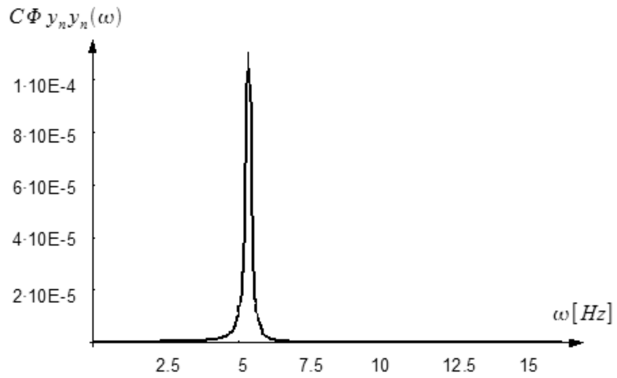

Fig. 9. The spectral density function for $a = 0.5$ ($v = 121$ km/h)

4. CONCLUSION

The paper presents the spectral analysis of the beam's vibration under a random train of moving forces which forms a filtered Poisson process. In this case, it is impossible to obtain a solution for the spectral density function for the system response in the classical way like for the excitation by a weakly stationary stochastic process. To overcome these difficulties the dynamic moving influence function has been introduced, which allows one to obtain the spectral density function of the normal mode response of the beam in a short expression (19). The spectral analysis allows one to determine the velocity of the moving forces for which the beam response is a narrow-band stochastic process. This is important from a practical point of view because the narrow-band stochastic response, where one frequency is dominating, is one of the factors which accelerate fatigue in the bridge structures.

ACKNOWLEDGEMENT

The authors want to thank Wojciech Zielichowski-Haber for performing the numerical calculations.

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