CONSTITUTIVE MODELS
FOR PREDICTING LIQUEFACTION OF SOILS

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Abtract: Liquefaction is an example of a diffuse mode of failure. It occurs in loose sands when an effective mean pressure falls to zero. This phenomenon has been studied extensively both experimentally and theoretically. Three constitutive laws, based on different assumptions, capable of predicting liquefaction are given. These are the Pastor–Zienkiewicz generalized plasticity model and Darve's incrementally non-linear and octo-linear models. The results of numerical simulations of element tests are presented.

1. INTRODUCTION

Liquefaction of soils can be defined as a state of vanishing intergranular stresses. This phenomenon may be caused by cyclic or static (monotonically increasing) loading (KRAMER and SEED [1]). Most often liquefaction occurs during earthquakes or other quasi-seismic events. The static liquefaction is the liquefaction resulting from the application of noncyclic shear stresses. Due to liquefaction failure can occur, spreading over large mass of soil deposits, commonly referred to as flow slides. This is an example of a diffuse mode of failure.

Liquefaction occurs in loose sands and is caused by decreasing effective stresses due to direct application of external forces (human activity, earthquakes) or indirectly through the changes of pore pressures (rainfalls). It can occur when the hydraulic gradient of an upward current in saturated soil or an upward gas current equalizes the gravity forces ("boiling sands") due to vibrations (earthquakes) of saturated loose or even medium dense sands or when rapid deviatoric loading is applied to a saturated very loose sand ("quicksand").

In the past decades, liquefaction of soils has been extensively studied both experimentally and theoretically. This short study presents three models capable of describing static liquefaction.

2. CONSTITUTIVE MODELS DESCRIBING LIQUEFACTION

2.1. INTRODUCTION

In general, constitutive laws are formulated by means of mathematical equations involving coefficients which represent parameters of a given material (a soil). Usually these coefficients are not constant but depend on the stress level and a state of soil.

Deformations of soils are irreversible with the effect of very low loading applied. Thus the relationship between stresses and strains must be non-linear. Moreover, a small load increment applied in a time increment induces a small unique response. This feature is known as the principle of determinism in the small and implies the incremental formulation of constitutive laws. This means that there must exist a tensorial function $\mathbf{H}$ relating strain, stress and time increments as well as scalar and/or tensorial internal variables (also called memory parameters):

$$\mathbf{H}_{\chi}(d\boldsymbol{\varepsilon}, d\boldsymbol{\sigma}, dt) = 0 . \tag{1}$$

For the materials insensitive to the rate of loading the constitutive function $\mathbf{H}_{\chi}$ is independent of the time increment dt. The constitutive ralationship can be expressed in the equivalent form (DARVE and ROGUIEZ [5]):

$$d\boldsymbol{\varepsilon} = \mathbf{G}_{\chi}(d\boldsymbol{\sigma}) \quad \text{or} \quad d\boldsymbol{\sigma} = \mathbf{G}_{\chi}^{-1}(d\boldsymbol{\varepsilon}) . \tag{2}$$

For inviscid materials the functions $\mathbf{G}_{\chi}$ and $\mathbf{G}_{\chi}^{-1}$ are homogeneous of degree 1, non-linear and anisotropic with respect to $d\boldsymbol{\sigma}$. For homogeneous functions Euler's theorem holds which gives:

$$
\begin{aligned}
d\boldsymbol{\varepsilon} &= \frac{\partial G_{\chi}}{\partial (d\boldsymbol{\sigma})} d\boldsymbol{\sigma} = \mathbf{M}_{\chi}\left(\frac{d\boldsymbol{\sigma}}{\|d\boldsymbol{\sigma}\|}\right) d\boldsymbol{\sigma}, \\
d\boldsymbol{\sigma} &= \frac{\partial G_{\chi}^{-1}}{\partial (d\boldsymbol{\varepsilon})} d\boldsymbol{\varepsilon} = \mathbf{P}_{\chi}\left(\frac{d\boldsymbol{\varepsilon}}{\|d\boldsymbol{\varepsilon}\|}\right) d\boldsymbol{\varepsilon},
\end{aligned}
\tag{3}
$$

where $\|\bullet\|$ denotes a norm in the stress or strain space.

The tensors $\mathbf{M}_{\chi}$ and $\mathbf{P}_{\chi}$ depend only on the internal variables χ and the direction of $d\boldsymbol{\sigma}$ or $d\boldsymbol{\varepsilon}$. This directional dependence of the constitutive tensors implies the existance of the so-called tensorial zones (DARVE and LABANIEH [4]). A tensorial zone is a domain in the loading space characterized by a unique tensor $\mathbf{M}_{\chi}$ ($\mathbf{P}_{\chi}$). Within the same tensorial zone the relationship between $d\boldsymbol{\sigma}$ and $d\boldsymbol{\varepsilon}$ is incrementally linear. The whole space of possible loading directions can be divided into a number of tensorial zones. All constitutive laws can be classified in terms of this number.

Elastoplastic constitutive laws with only one unique criterion distinguishing between loading and unloading are the examples of laws with two tensorial zones. In particular, the models proposed by Pastor and Zienkiewicz (PASTOR et. al. [6]) formulated within the framework of generalized plasticity theory represent this group.

The models proposed by Darve are formulated in a different way. An incrementally octo-linear model (DARVE and LABANIEH [4]) is the example of the law with eight tensorial zones, whereas the incrementally non-linear model (DARVE [2]) represents the group with infinite tensorial zones.

All the models mentioned can predict the liquefaction of sands.

2.2. GENERALIZED PLASTICITY

Generalized plasticity concept introduced by ZIENKIEWICZ and MRÓZ [8] and developed by ZIENKIEWICZ et al. [9] and PASTOR et al. [6], [7] defines in the stress space a unit vector **n** which determines loading and unloading directions for any stress state. Strain increments are given as follows:

$$d\boldsymbol{\varepsilon} = \mathbf{C}^L d\boldsymbol{\sigma} \quad \text{for} \quad d\boldsymbol{\sigma}^T \mathbf{n} > 0 \ \text{(loading)}, \tag{4}$$

$$d\boldsymbol{\varepsilon} = \mathbf{C}^U d\boldsymbol{\sigma} \quad \text{for} \quad d\boldsymbol{\sigma}^T \mathbf{n} < 0 \ \text{(unloading)}. \tag{5}$$

The case of neutral loading, when no irreversible strain occurs, is defined by:

$$d\boldsymbol{\sigma}^T \mathbf{n} = 0 . \tag{6}$$

The continuity condition between loading and unloading implies the general form of the constitutive tensors:

$$\mathbf{C}^{L/U} = \mathbf{C}^e + \frac{1}{H_{L/U}} \mathbf{n}_{gL/U} \mathbf{n}^T , \tag{7}$$

where $\mathbf{n}_{gL/U}$ are the arbitrary tensors of unit norm and $H_{L/U}$ are the tangent plastic moduli (the scalar functions of loading and unloading parameters).

When neutral loading takes place a strain increment is uniquely determined since:

$$\forall d\boldsymbol{\sigma} : d\boldsymbol{\sigma}^T \mathbf{n} = \mathbf{0}, \quad \mathbf{C}^L d\boldsymbol{\sigma} = \mathbf{C}^U d\boldsymbol{\sigma} \Rightarrow d\boldsymbol{\varepsilon} = \mathbf{C}^e d\boldsymbol{\sigma} . \tag{8}$$

The tensor $\mathbf{C}^e$ characterizes the elastic behaviour of a soil, since during neutral loading no irreversible strain is produced.

It is assumed that the strain increment can be divided into two components: elastic and plastic:

$$d\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon}^e + d\boldsymbol{\varepsilon}^p , \tag{9}$$

where

$$d\mathbf{\varepsilon}^e = \mathbf{C}^e d\mathbf{\sigma} , \qquad d\mathbf{\varepsilon}^p = \frac{1}{H_{L/U}} \mathbf{n}_{gL/U} \mathbf{n}^T d\mathbf{\sigma} . \tag{10}$$

In the generalized plasticity theory, particular components ($\mathbf{n}_{gL/U}$, $\mathbf{n}$, $H_{L/U}$) can be postulated without introducing the notions of yield surface nor plastic potential.

As observed in experiments, loose sands loaded under undrained conditions exhibit loss of the effective intergranular stress (the mean stress in figure 1a) and a peak in deviatoric stress (figure 1b) after which the strength reduces to zero and the pore pressure increases continuously (figure 1c). The important thing is that the pore pressure increases during the whole loading. This means that sand densifies which indicates hardening.
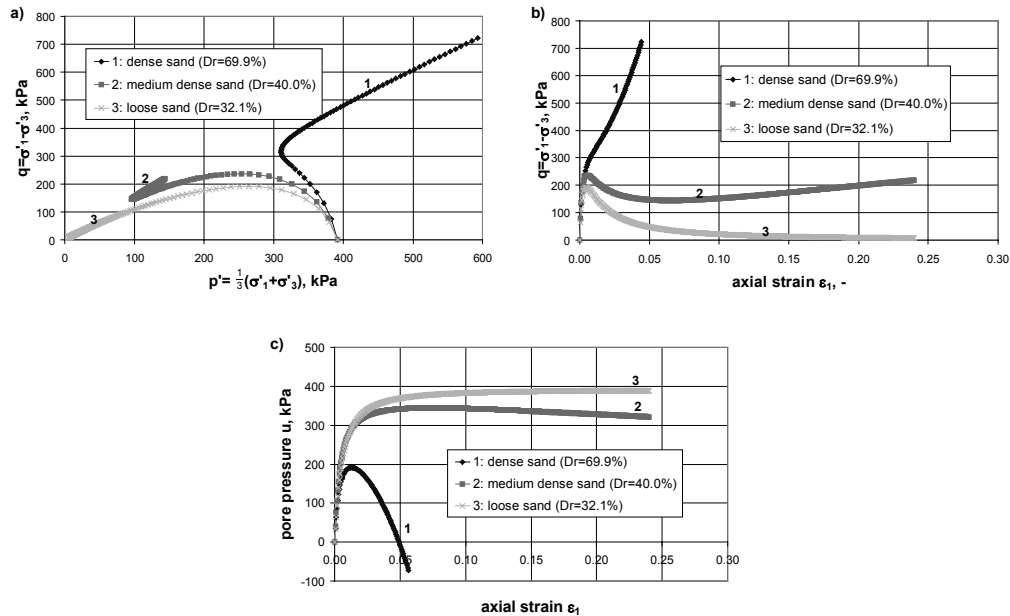

Fig. 1. Predictions of undrained sand behaviour according to
the Pastor–Zienkiewicz model

In order to simulate a decrease in strength in hardening regime, non-associated flow rule must be assumed. In the model proposed by Pastor and Zienkiewicz for sands, this effect has been achieved by assuming two different sets of constants defining the unit tensors $\mathbf{n}_{gL}$ and $\mathbf{n}$ during loading ($M_{f/g}$, $\alpha_{f/g}$). The assumed formulae in triaxial stress space are as follows:

$$\mathbf{n}_{gL} = (n_{gL}^p \quad n_{gL}^q \quad n_{gL}^\theta)^T, \quad \mathbf{n} = (n^p \quad n^q \quad n^\theta)^T, \tag{11}$$

$$d_g = (1+\alpha_g)\left(M_g - \frac{q}{p'}\right), \quad d_f = (1+\alpha_f)\left(M_f - \frac{q}{p'}\right), \tag{12}$$

$$n_{gL}^p = \frac{d_g}{\sqrt{1+d_g^2}}, \quad n^p = \frac{d_f}{\sqrt{1+d_f^2}}, \tag{13}$$

$$n_{gL}^q = \frac{1}{\sqrt{1+d_g^2}}, \quad n^q = \frac{1}{\sqrt{1+d_f^2}}, \tag{14}$$

$$n_{gL}^\theta = -\frac{qM_g\cos 3\theta}{2\sqrt{1+d_g^2}}, \quad n^\theta = -\frac{qM_f\cos 3\theta}{2\sqrt{1+d_f^2}}, \tag{15}$$

where p' is the mean stress, q – the deviatoric stress, θ – the Lode's angle.

2.3. MODELS BASED ON INTERPOLATION RULES

Polynomial series expansion for the function M_{ijkl} (equation (3a)) gives:

$$M_{ijkl} = M_{ijkl}^0 + M_{ijklpq}^1 \frac{d\sigma_{pq}}{\sqrt{d\sigma_{pq}d\sigma_{pq}}} + M_{ijklpqmn}^2 \frac{d\sigma_{pq}}{\sqrt{d\sigma_{pq}d\sigma_{pq}}} \frac{d\sigma_{mn}}{\sqrt{d\sigma_{mn}d\sigma_{mn}}} + ... \tag{16}$$

Equations (3a) and (16) give:

$$d\varepsilon_{ij} = M_{ijkl}^0 d\sigma_{kl} + \frac{1}{\|d\boldsymbol{\sigma}\|} M_{ijklpq}^1 d\sigma_{kl}d\sigma_{pq} + \frac{1}{\|d\boldsymbol{\sigma}\|^2} M_{ijklpqmn}^2 d\sigma_{kl}d\sigma_{pq}d\sigma_{mn} + ... \tag{17}$$

Equation (17) is the general form of a family of rate-independent constitutive relations. The first term represents all the elastic (hypoelastic) laws. After rejecting terms of the order higher than two we obtain the *incrementally non-linear constitutive relations of second order* (DARVE [2]):

$$d\varepsilon_{ij} = M_{ijkl}^0 d\sigma_{kl} + \frac{1}{\|d\boldsymbol{\sigma}\|} M_{ijklpq}^1 d\sigma_{kl}d\sigma_{pq}. \tag{18}$$

Three assumptions have been made to complete the formulation of the model:
• the incremental relationship (18) is orthotropic, which means that the symmetries with respect to the planes of spatial coordinate system are postulated;
• the shear behaviour is incrementally linear, i.e., $M_{ijklpq}^1 = 0$ for $k, l, p, q \geq 4$;

• non-linearity is restricted to the diagonal terms of M^1_{ijklpq}, i.e. $M^1_{ijklpq} = 0$ for $(k, l) \neq (p, q)$.

Equation (18) takes the following form:

$$
\begin{pmatrix} d\varepsilon_{11} \\ d\varepsilon_{22} \\ d\varepsilon_{33} \end{pmatrix} = \frac{1}{2}\left[\mathbf{N}^+ + \mathbf{N}^-\right] \cdot \begin{pmatrix} d\sigma_{11} \\ d\sigma_{22} \\ d\sigma_{33} \end{pmatrix} + \frac{1}{2\|d\boldsymbol{\sigma}\|}\left[\mathbf{N}^+ - \mathbf{N}^-\right] \cdot \begin{pmatrix} d\sigma_{11}^2 \\ d\sigma_{22}^2 \\ d\sigma_{33}^2 \end{pmatrix},
$$
$$
(19)
$$
$$
d\varepsilon_{12} = \frac{1}{2G_3}d\sigma_{12}, \quad d\varepsilon_{23} = \frac{1}{2G_1}d\sigma_{23}, \quad d\varepsilon_{31} = \frac{1}{2G_2}d\sigma_{31},
$$

where:

$$
\mathbf{N}^+ = \begin{bmatrix} \dfrac{1}{E_1^+} & -\dfrac{v_2^{1+}}{E_2^+} & -\dfrac{v_3^{1+}}{E_3^+} \\[2mm] -\dfrac{v_1^{2+}}{E_1^+} & \dfrac{1}{E_2^+} & -\dfrac{v_3^{2+}}{E_3^+} \\[2mm] -\dfrac{v_1^{3+}}{E_1^+} & -\dfrac{v_2^{3+}}{E_2^+} & \dfrac{1}{E_3^+} \end{bmatrix}, \quad
\mathbf{N}^- = \begin{bmatrix} \dfrac{1}{E_1^-} & -\dfrac{v_2^{1-}}{E_2^-} & -\dfrac{v_3^{1-}}{E_3^-} \\[2mm] -\dfrac{v_1^{2-}}{E_1^-} & \dfrac{1}{E_2^-} & -\dfrac{v_3^{2-}}{E_3^-} \\[2mm] -\dfrac{v_1^{3-}}{E_1^-} & -\dfrac{v_2^{3-}}{E_2^-} & \dfrac{1}{E_3^-} \end{bmatrix} \qquad (20)
$$

with E_i and v_i^j – the Young modulus and Poisson's ratio, respectively. The superscript "+" refers to compression ($d\sigma_{ii} > 0$), and "–" to extension ($d\sigma_{ii} < 0$) with respect to the direction i of "generalized" triaxial paths (the other two stresses are constant: $d\sigma_{jj} = d\sigma_{kk} = 0$).

The "generalized" tangent Young moduli and Poisson's ratios are defined along these paths by (no summation):

$$
E_i = \left.\frac{\partial \sigma_{ii}}{\partial \varepsilon_{ii}}\right|_{\sigma_{jj}, \sigma_{kk}}, \quad v_i^j = -\left.\frac{\partial \varepsilon_{jj}}{\partial \varepsilon_{ii}}\right|_{\sigma_{jj}, \sigma_{kk}}, \quad v_i^k = -\left.\frac{\partial \varepsilon_{kk}}{\partial \varepsilon_{ii}}\right|_{\sigma_{jj}, \sigma_{kk}}. \qquad (21)
$$

Equations (19) are homogeneous of degree one with respect to $d\boldsymbol{\sigma}$. This means that they describe a rate-independent behaviour. They are also non-linear with respect to the stress increment (regarding square terms), which means that they can describe plastic (irreversible) strains after a stress cycle ($d\sigma, -d\sigma$). This means that it is impossible to decompose a total strain increment into an elastic and plastic parts. Elasticity and plasticity are intrinsically mixed within the constitutive relation.

The set of equations (19) is a non-linear (quadratic) interpolation between the material responses to generalized triaxial loading. Making use of the same matrices $\mathbf{N}^{\pm}$ (equation (20)) linear interpolation between orthogonal directions can be made giving the "octo-linear" model (DARVE and LABANIEH [4]):

$$\begin{pmatrix} d\varepsilon_{11} \\ d\varepsilon_{22} \\ d\varepsilon_{33} \end{pmatrix} = \frac{1}{2} \left[\mathbf{N}^+ + \mathbf{N}^- \right] \cdot \begin{pmatrix} d\sigma_{11} \\ d\sigma_{22} \\ d\sigma_{33} \end{pmatrix} + \frac{1}{2\|d\sigma\|} \left[\mathbf{N}^+ - \mathbf{N}^- \right] \cdot \begin{pmatrix} |d\sigma_{11}| \\ |d\sigma_{22}| \\ |d\sigma_{33}| \end{pmatrix}. \tag{22}$$

For both models the behaviour of a soil has to be checked in triaxial tests (preferably true triaxial) and described by analytical functions that involve state variables and memory parameters. Thus, these quantities are implicitly contained in the matrices $\mathbf{N}^{\pm}$.

3. LIQUEFACTION CRITERION

It is commonly stated that the water saturation is one of the necessary conditions to initiate liquefaction. The results obtained by Lanier and Bloch (DARVE [3]) prove the possibility of liquefaction of dry sand. So the deciding factor is the lack of volume variation or at least its limitation.
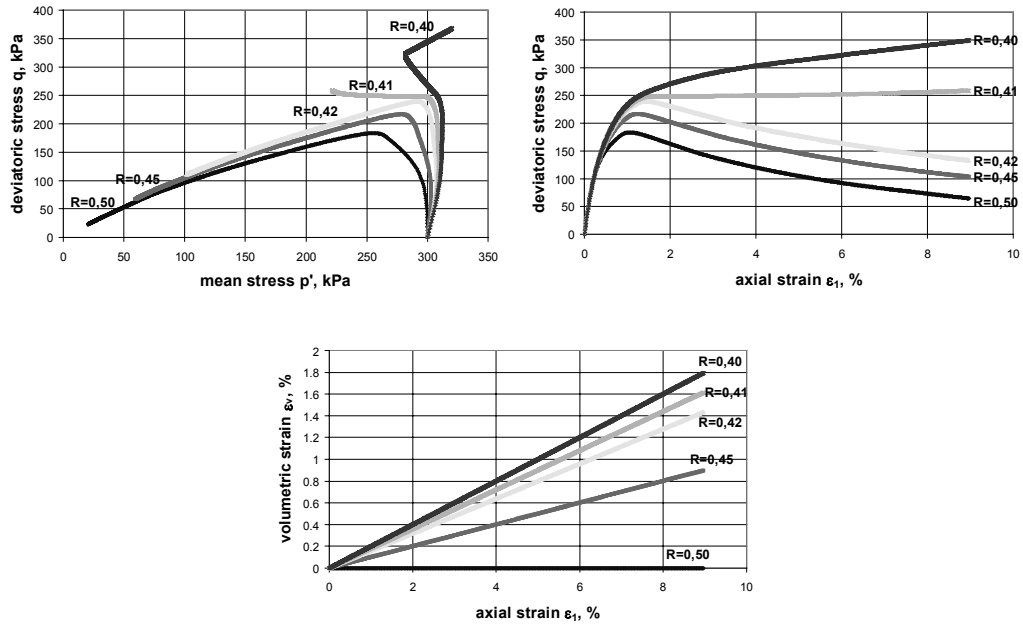

Fig. 2. Simulation of proportional strain paths under different isochoric conditions

In general, under triaxial conditions the proportional strain loading is given by:

$$d\varepsilon_1 = \text{const}, \quad d\varepsilon_2 = d\varepsilon_3 = -R d\varepsilon_1. \tag{23}$$

R is the constant representing the degree of limitation of volumetric variation. $R = 0.5$ indicates isochoric loading (no variation of volume).

Figure 2 presents the results of the simulations of triaxial loading at different values of *R* performed on loose sand described by the incrementally non-linear law. In these simulations, liquefaction occurs for $R > 0.41$.

4. CONCLUSIONS

Static liquefaction occurs in loose sands loaded under approximately isochoric conditions. This may lead to dangerous phenomena such as landslides. Liquefaction can be predicted based on constitutive laws of different classes. The capability of liquefaction prediction depends mainly on the ability to reproduce the non-linear inelastic and non-associative behaviour of a soil.

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