Studia Geotechnica et Mechanica, Vol. XXX, No. 3–4, 2008

A THREE-DIMENSIONAL MICROMECHANICAL MODELLING OF THE ANISOTROPY OF GRANULAR MEDIA

JAMILA RAHMOUN*

Laboratory of Industrial and Human Automatics, Mechanics and Informatics, UMR CNRS 8530, University of Valenciennes, Le Mont Houy, 59313 Valenciennes, France.

OLIVIER MILLET

Laboratoire d'Etude des Phénomènes de Transfert Appliqués au Bâtiment, Université de La Rochelle, avenue Michel Crepeau, 17000 La Rochelle, France.

DJIMÉDO KONDO

Laboratoire de Mecanique de Lille, UMR CNRS 8107, Université de Lille, Bd. Paul Langevin, 59655 Villeneuve d'Ascq, France.

Abstract: This paper presents a micromechanical approach to the granular media with a particular account of the texture-induced anisotropy. The procedure is based on the use of a second-order fabric tensor corresponding to the anisotropy induced by the distribution of contacts. Incorporation of this fabric tensor into a homogenization scheme allows us to obtain the anisotropic elastic properties of the material. Besides the classical Voigt localization, as well as the Reuss approach, we propose a new kinematics-based localization rule which generalizes the one provided by [4] in the case of an isotropic contact distribution. Finally, it is demonstrated that the results deduced from the proposed localization rule agree with the numerical result obtained [10] by means of discrete numerical simulations.

Streszczenie: Przedstawiono mikromechaniczne podejście do ośrodków ziarnistych ze szczególnym uwzględnieniem anizotropii spowodowanej przez ich teksturę. Podejście to opiera się na użyciu tensora drugiego rzędu, który odpowiada anizotropii wywołanej rozkładem styków. Dzięki włączeniu tego tensora do schematu homogenizacji otrzymuje się anizotropowe i sprężyste właściwości materiału. Oprócz klasycznej lokalizacji Voigta i podejścia Reussa zaproponowano nową, opartą na kinematyce, regułę lokalizacji, która uogólnia reguły podane w [4] w przypadku rozkładu izotropowych kontaktów. Na koniec pokazano, że wyniki wydedukowane na podstawie zaproponowanej reguły lokalizacji [10] są zgodne z wynikami dyskretnych symulacji numerycznych.

Резюме: Представлен микромеханический подход к зернистым средам с особенным учетом анизотропии, вызванной их текстурой. Этот подход базирует на употреблении тензора второго порядка, который отвечает анизотропии, вызванной расположением стыков. Благодаря включению этого тензора в схему гомогенизации, получаются анизотропные и упругие свойства материала. Кроме классического распределения Войта и подхода Рейсса, было предложено новое, базирующее на кинематике, правило распределения, которое обобщает правила данные

* Corresponding author: tel.: 0033320337166, fax: 0033320337153, e-mail: jamila.rahmoun@univ-valenciennes.fr

в [4] в случае распределнния изотропных контактов. Наконец было показано, что результаты, выведенные на основе предложенного правила распределения [10] согласны результатам дискретных численных имитаций.

1. INTRODUCTION

The average behaviour of a grains' assembly is the result of the particle interactions due to contacts. The relevant microscopic variables are strongly related to the contact forces between grains and to the relative displacement of grains. Whatever is the mechanical nature of these contacts, their geometrical network implies a particular structure which constitutes a fundamental component of the granular media. This structure corresponds to the texture of the medium which can be influenced by different factors [2], [8], [21], [22], [28], [34].

The first factor is the shape of grains. Indeed, for non-spherical grains, but with prolate or oblate shape, one has a preferential orientation of grains in the assembly and consequently a directional character of the contact distribution. The methodology of the preparation of the sample of a granular material (deposit of grains by a source point or by pluviation, ...) and the deformations history may also strongly affect the distribution of the contacts [24], [25], [29], [30].

Since there is an obvious link between the contact direction and the anisotropy of the granular assembly referring to its contact network, it is particularly interesting to investigate the possibility of predicting the macroscopic anisotropic properties from the microstructure. This is precisely the main purpose of homogenization procedure (see [4], [5], [7], [11]).

This paper is mainly devoted to the description of the grains' contact distribution by using various degrees of the fabric tensor. In order to simplify the problem, we assume that at the contact between grains, the local relation between forces and relative disques is linear. We then investigate a new kinematical rule which provides the link between microscopic and macroscopic strain fields. Comparison between the obtained closed-form expression of the elastic stiffness and numerical results shows how the homogenization procedure is interesting.

The following tensorial notations are adopted:

$$(\underline{\underline{a}} \otimes \underline{\underline{b}})_{ijkl} = a_{ij} b_{kl}; \qquad (\underline{\underline{a}} \,\overline{\otimes}\, \underline{\underline{b}}) = \frac{1}{2}(\underline{\underline{a}} \otimes \underline{\underline{b}} + \underline{\underline{a}} \,\overline{\otimes}\, \underline{\underline{b}}).$$

2. DESCRIPTION OF THE TEXTURE OF THE GRANULAR MATERIAL

It is well-known that the contact probability $P(\underline{n})$ in the normal direction $\underline{n}$ is most often used to define the fabric tensor of granular media [33]. Classically, the second-order tensorial representation of the fabric tensor reads in the *3D* (see for instance [21], [27], [31]):

$$\underline{\underline{D}} = \frac{1}{4\pi} \int_{s^2} P(\underline{n})(\underline{n} \otimes \underline{n}) ds \,, \tag{1}$$

where N is the total number of grains, and $P(\underline{n})$ is the distribution of the contact probability. It satisfies the following normalization condition:

$$\frac{1}{4\pi} \int_{s^2} P(\underline{n}) ds = 1 \tag{2}$$

for the anisotropic medium. In the case of an isotropic one, it is clear that $P(\underline{n}) = 1$.

Combining equation (1) with relation (2) leads to the following expression of the contact distribution (see the appendix B.1) [27], [31]:

$$P(\underline{n}) = \frac{15}{2} \left[\underline{\underline{D}} : (\underline{n} \otimes \underline{n}) - \frac{1}{5} \right]. \tag{3}$$

It must be emphasized that the description of the contact distribution by the second-order fabric tensor $\underline{\underline{D}}$ is sufficient when the material symmetry is orthotropic.

In order to highlight this anisotropic phenomenon, we propose the discrete simulation of the granular material compaction under gravity. This simulation has been performed by means of MULTICOR software developed by FORTIN [17] (see figure 1).

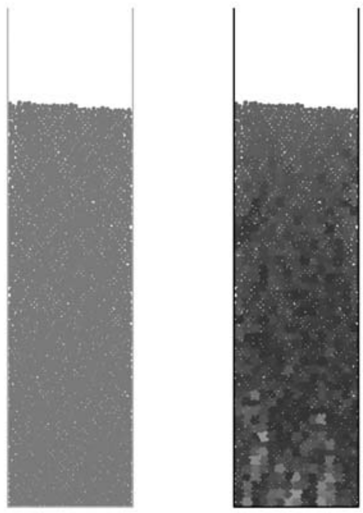

Fig. 1. Compaction under gravity of an ensiled analogical material

The distribution of the contact given by this simulation is shown in figure 2. Then, by discretizing the interval [0, π] in 20 intervals:

$$I_i = \left[i\frac{\pi}{20}, (i+1)\frac{\pi}{20} \right], \quad i = 0.19\,,$$

we determine the orientation contact angle θ which defines the normal vector $\underline{n}$ at this point of contact. We compute the number of contacts $N(i)$ corresponding to each interval I_i of orientation and then the distribution of the contact:

$$n(i) = \frac{N(i)}{k}\,,$$

where k is the total number of contacts.

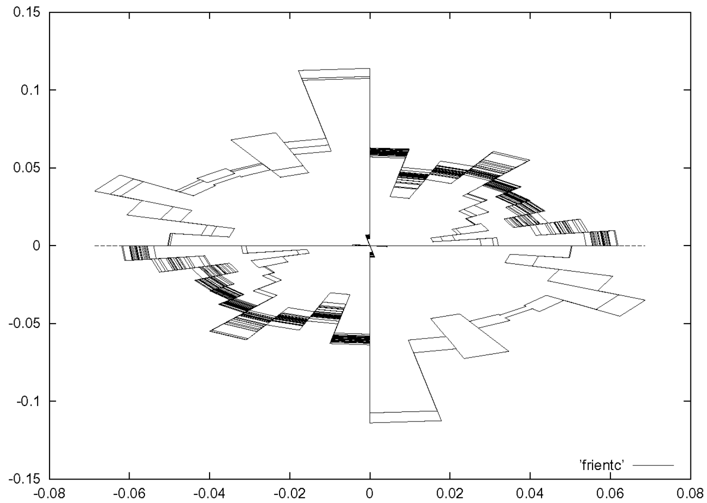

Fig. 2. The contact orientation distributions

Figure 2 shows a polar diagram of the contacts which indicates a strong anisotropy of the contacts' network.

3. RESULTS OBTAINED BY MEANS OF THE ASSUMPTION OF VOIGT

3.1. PRINCIPLE AND METHODOLOGY

The kinematic assumption of Voigt, which assumes that the deformations are uniform inside a representative element volume (R.E.V), provides a simple relation between the relative displacements of grains and the macroscopic deformations [36], [8], [1]. It is recalled that the kinematic assumption of Voigt classically writes [16]:

$$u_i^c = E_{ij} L_j^c \,, \tag{4}$$

where E_{ij} are the components of the macroscopic strain tensor and L_j^c is the branch vector which joins the grains centers in the contact point *c*.

Now, we aim at building, by homogenization, the macroscopic elastic behaviour of the granular medium with account of anisotropy effects due to the contact distribution. A linear elastic contact law is considered. Denoting by F^c the contact forces between grains, we arrive at a classical definition of the quasi-static average stresses tensor [14], [26], [32], [37]:

$$\underline{\underline{\Sigma}} = \frac{1}{V} \sum_{c=1}^{N} \underline{F}^c \otimes \underline{L}^c \,, \tag{5}$$

where V designates the volume of the R.E.V considered, and where the summation is performed on the N interior contacts. From (5), it appears that the macroscopic stresses' tensor depends on the distribution of the local contact forces. However, $\underline{\underline{\Sigma}}$ as defining by (5) is not symmetric; therefore, another definition established by a direct use of the principle of the virtual work [6] is generally adopted. Indeed:

$$\underline{\underline{\Sigma}} : \Delta \underline{\underline{E}} = \frac{1}{V} \sum_{c=1}^{N} (\underline{F}^c . \Delta \underline{u}^c) \,. \tag{6}$$

where $\Delta \underline{u}^c$ designates the virtual displacements of grains associated with a virtual macroscopic deformation $\underline{\underline{E}}$. Again, the summation is considered only on the active contacts $c = 1$, ..., N. Considering now the Voigt assumption (6), we obtain:

$$\underline{\underline{\Sigma}} : \Delta \underline{\underline{E}} = \frac{1}{V} \sum_{c=1}^{N} (\underline{F}^c \otimes \underline{L}^c) : \Delta \underline{\underline{E}} \,. \tag{7}$$

Since $\Delta \underline{\underline{E}}$ is symmetric, only the symmetric part of $\frac{1}{V} \sum_{c=1}^{N} (\underline{F}^c \otimes \underline{L}^c)$ contributes to (7). In consequence, a suitable definition of the average stresses tensor reads:

$$\underline{\underline{\Sigma}} = \frac{1}{V} \left[\sum_{c=1}^{N} \underline{F}^c \otimes \underline{L}^c \right]^{\mathrm{sym}} , \tag{8}$$

where $(\underline{\underline{a}})^{\mathrm{sym}} = \frac{1}{2} (\underline{\underline{a}} + \underline{\underline{a}}^T)$.

It must be emphasized that in the case of a great number of grains, the antisymmetrical part of expression (5) is negligible [13], and (5) and (8) lead to the same results.

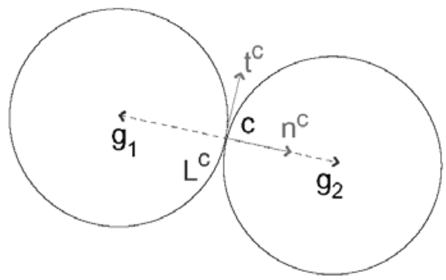

Fig. 3. Representative scheme of the contact between two grains

For simplicity, we suppose in this study that the granular assembly is composed of spherical grains with the same radius r. We have therefore $\underline{L}^c = \| \underline{L}^c \| \underline{n}^c = 2r\underline{n}^c$ (see figure 3). It follows that the Voigt kinematic assumption reads:

$$u_i^c = E_{ij} L_j^c = 2rE_{ij}\underline{n}_j^c = 2r(\underline{\underline{E}}.\underline{n}^c)_i \,. \tag{9}$$

Let us now introduce the following decomposition of the displacement $\underline{u}^c$ in the local basis in the contact (t, n):

$$\underline{u}^c = u_n^c \underline{n}^c + \underline{u}_t^c \,.$$

From (9), it is readily seen that:

$$\begin{aligned} u_n^c &= \underline{u}^c.\underline{n}^c = 2r\underline{n}^c.\underline{\underline{E}}.\underline{n}^c = 2r(\underline{n}^c \otimes \underline{n}^c) : \underline{\underline{E}} \,, \\ u_t^c &= u^c - u_n^c \underline{n}^c = 2r\underline{\underline{E}}.\underline{n}^c - u_n^c \underline{n}^c = 2r\underline{\underline{\underline{T}}}^c : \underline{\underline{E}} \,, \end{aligned} \tag{10}$$

with

$$\underline{\underline{\underline{T}}}^c = \underline{n}^c.\mathbb{I} = -(\underline{n}^c \otimes \underline{n}^c \otimes \underline{n}^c)$$

in which $\mathbb{I}$ denotes the symmetric fourth-order unit tensor:

$$I_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \,.$$

As already indicated, a linear contact law is considered:

$$\underline{F}^c = K_n u_n^c \underline{n}^c + K_t \underline{u}_t^c \,, \tag{11}$$

$\underline{n}^c$ is the normal unit vector in the contact. Its inverse takes the form:

$$\underline{u}^c = H_n F_n^c \underline{n}^c + H_t \underline{F}_t^c\,, \tag{12}$$

$\underline{F}^c$ and $\underline{u}^c$ denote, respectively, the contact forces and the relative displacements' vector in the contact. Finally, K_t and H_t (respectively K_n and H_n) denote the tangential rigidities and flexibilities (respectively normal) in the contact.

We aim now at establishing an expression of the homogenized elastic stiffness tensor $\mathbb{C}^{\text{hom}}$. The approach consists first in establishing a relation between $\underline{\underline{\Sigma}}$ and $\underline{\underline{E}}$ by combining (8), (11) and (4). Consideration of (11) in (4) gives:

$$\underline{\underline{\Sigma}} = \frac{1}{V}\left[\sum_{c=1}^{N}(\underline{F}^c \otimes \underline{L}^c)\right]^{\text{sym}} = \frac{2r}{V}\left[\sum_{c=1}^{N}(F_n^c \underline{n}^c + \underline{F}_t^c)\otimes \underline{n}^c\right]^{\text{sym}}, \tag{13}$$

where the summation is carried out on the active contacts $c = 1, ..., N$.

On the other hand, it can be easily observed that:

$$(\underline{F}_t^c \otimes \underline{n}^c)^{sym} = \underline{\underline{T}}^{cT}.\underline{F}_{\text{t}}^{\text{c}}\,.$$

Indeed:

$$\begin{aligned}
(\underline{\underline{T}}^{cT}.\underline{F}_t^c)_{ij} &= [(\underline{n}^c.\mathbb{I} - \underline{n}^c\otimes\underline{n}^c\otimes\underline{n}^c)^T.\underline{F}_t^c]_{ij} = [(\mathbb{I}.\underline{n}^c - \underline{n}^c\otimes\underline{n}^c\otimes\underline{n}^c)^T.\underline{F}_t^c]_{ij} \\
&= [(\mathbb{I}.\underline{n}^c).\underline{F}_t^c]_{ij} = I_{ijkl} n_l^c (F_t)_k^c \\
&= \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) n_l^c (F_t)_k^c = \frac{1}{2}[n_j^c (F_t)_i^c + n_i^c (F_t)_j^c]
\end{aligned}$$

in which we have made use of the fact that $\underline{n}^c.\underline{F}_t^c = 0$. Using this result in (13), we obtain:

$$\underline{\underline{\Sigma}} = \frac{2r}{V}\sum_{c=1}^{N}\ [F_n^c \underline{n}^c \otimes \underline{n}^c + \underline{\underline{T}}^{cT}\cdot \underline{F}_T^c]\,. \tag{14}$$

Considering the contact law (11) and the kinematic assumption, we arrive at:

$$\begin{aligned}
F_n^c &= K_n u_n^c = 2rK_n(\underline{n}^c\otimes\underline{n}^c):\underline{\underline{E}}, \\
F_t^c &= K_t u_t^c = 2rK_t \underline{\underline{T}}^c : \underline{\underline{E}}
\end{aligned} \tag{15}$$

which when inserted in (14) leads to:

$$\underline{\underline{\Sigma}} = \frac{4r^2}{V}\sum_{c=1}^{N}\ [K_n(\underline{n}^c\otimes\underline{n}^c\otimes\underline{n}^c\otimes\underline{n}^c) + K_t(\underline{\underline{T}}^{cT}.\underline{\underline{T}}^c)]:\underline{\underline{E}}\,. \tag{16}$$

It immediately follows that the macroscopic elastic behaviour:

$$\underline{\underline{\Sigma}} = \mathbb{C}^{\text{hom}} : \underline{\underline{E}} \tag{17}$$

is given by the following expression of the homogenized elastic stiffness tensor:

$$\mathbb{C}^{\text{hom}} = \frac{4r^2}{V} \sum_{c=1}^{N} \left[K_n (\underline{n}^c \otimes \underline{n}^c \otimes \underline{n}^c \otimes \underline{n}^c) + K_t (\underline{\underline{T}}^{cT} . \underline{\underline{T}}^c) \right]. \tag{18}$$

Now, it is possible to model the elastic behaviour of granular assembly by taking into account the contacts' distribution. In terms of the probability of the contacts $P(\underline{n})$, $\mathbb{C}^{\text{hom}}$ reads:

$$\mathbb{C}^{\text{hom}} = \frac{4Nr^2}{V} \left[K_n \frac{1}{4\pi} \int_{s^2} P(\underline{n})(\underline{n} \otimes \underline{n} \otimes \underline{n} \otimes \underline{n}) ds + K_t \frac{1}{4\pi} \int_{s^2} P(\underline{n})(\underline{\underline{T}}^T . \underline{\underline{T}}) ds \right]. \tag{19}$$

3.2. PREDICTION BASED ON THE SECOND-ORDER FABRIC TENSOR

We take advantage of the fact that the contact distribution probability $P(\underline{n})$ can be defined from the second fabric tensor in order to express $\mathbb{C}^{\text{hom}}$ by means of the fabric tensor $\underline{\underline{D}}$. To do this, we first observed that

$$\frac{1}{4\pi} \int_{s^2} P(\underline{n})(\underline{n} \otimes \underline{n} \otimes \underline{n} \otimes \underline{n}) ds \quad \text{and} \quad \frac{1}{4\pi} \int_{s^2} P(\underline{n})(\underline{\underline{T}}^T . \underline{\underline{T}}) ds .$$

Replacing these quantities by their expression (51) and (52) derived in the appendix, we have:

$$\begin{aligned}\mathbb{C}^{\text{hom}} = \frac{4Nr^2}{V} \Bigg\{ & \frac{1}{7} K_n \left[-\frac{1}{5} (\underline{\underline{\delta}} \otimes \underline{\underline{\delta}} + 2 \underline{\underline{\delta}} \overline{\overline{\otimes}} \underline{\underline{\delta}}) + (\underline{\underline{D}} \otimes \underline{\underline{\delta}} + \underline{\underline{\delta}} \otimes \underline{\underline{D}}) + 2 (\underline{\underline{D}} \overline{\overline{\otimes}} \underline{\underline{\delta}} + \underline{\underline{\delta}} \overline{\overline{\otimes}} \underline{\underline{D}} \right] \\ & - \frac{1}{7} K_t \left[-\frac{1}{5} (\underline{\underline{\delta}} \otimes \underline{\underline{\delta}} + 2 \underline{\underline{\delta}} \overline{\overline{\otimes}} \underline{\underline{\delta}}) + (\underline{\underline{D}} \otimes \underline{\underline{\delta}} + \underline{\underline{\delta}} \otimes \underline{\underline{D}}) - \frac{3}{2} (\underline{\underline{D}} \overline{\overline{\otimes}} \underline{\underline{\delta}} + \underline{\underline{\delta}} \overline{\overline{\otimes}} \underline{\underline{D}} \right] \Bigg\}\end{aligned} \tag{20}$$

which satisfies the usual symmetries of the elasticity tensor. In the general case, $\mathbb{C}^{\text{hom}}$ has the standard symmetries (21 independent coefficients). Its components, whose detailed expressions are given in appendix B.4, depend on the components of $\underline{\underline{D}}$ and on the contact rigidities K_n and K_t.

3.3. ISOTROPIC DISTRIBUTION OF CONTACTS

As a simple illustration, we consider now an isotropic distribution of the contacts. In this case, $P(\underline{n}) = 1$ and:

$$\underline{\underline{D}} = \frac{1}{4\pi}\int_{s^2} \underline{n}\otimes\underline{n}ds = \frac{1}{3}\underline{\underline{\delta}} ,$$

$$\mathbb{D} = \frac{1}{4\pi}\int_{s^2} \underline{n}\otimes\underline{n}\otimes\underline{n}\otimes\underline{n}ds = \frac{1}{15}(\underline{\underline{\delta}}\otimes\underline{\underline{\delta}} + 2\underline{\underline{\delta}}\,\overline{\underline{\underline{\otimes}}}\,\underline{\underline{\delta}}) . \tag{21}$$

The prediction (20) corresponding to the second-order fabric tensor reads:

$$\mathbb{C}^{\mathrm{hom}} = \frac{4Nr^2}{15V}\{K_n(\underline{\underline{\delta}}\otimes\underline{\underline{\delta}} + 2\underline{\underline{\delta}}\,\overline{\underline{\underline{\otimes}}}\,\underline{\underline{\delta}}) - K_t(\underline{\underline{\delta}}\otimes\underline{\underline{\delta}} + 3\underline{\underline{\delta}}\,\overline{\underline{\underline{\otimes}}}\,\underline{\underline{\delta}})\} ,$$

from which the macroscopic compressibility k^{hom} and the macroscopic shear moduli μ^{hom} are obtained:

$$\mathbb{C}^{\mathrm{hom}} = k^{\mathrm{hom}}(\underline{\underline{\delta}}\otimes\underline{\underline{\delta}}) + 2\mu^{\mathrm{hom}}(\underline{\underline{\delta}}\,\overline{\underline{\underline{\otimes}}}\,\underline{\underline{\delta}}) ,$$

$$\mu^{\mathrm{hom}} = \frac{2}{15}\frac{Nr^2k_n}{V}(2+3\alpha) \quad \text{and} \quad k^{\mathrm{hom}} = \frac{4}{9}\frac{Nr^2k_n}{V} , \tag{22}$$

where $\alpha = K_t/K_n$. This result coincides with the one that is established in [10] describing only the isotropic case (see also [36], [9], [19]). This confirms the coherence of our approach developed in a more general context.

4. THE REUSS ASSUMPTION

4.1. BASIC PRINCIPLE

By using the Voigt assumption, we have easily obtained a law of the overall behaviour of the granular medium. However, this assumption constraines the particles' motion and constitutes a very particular solution of the problem [11], [12], [23].

In this section, we propose to build another homogenization scheme based on the static localization assumption of Reuss. This localization rule relies on the contact forces between grains and the macroscopic stress $\underline{\underline{\Sigma}}$ as follows [4], [15]:

$$\underline{F}^c = \underline{\underline{\Sigma}}.\underline{\underline{A}}.\underline{n}^c , \tag{23}$$

where $\underline{\underline{A}}$ is a symmetric second-order tensor which plays the role of the localization tensor.

By inserting (23) into (8) and by using the description in term of the contact probability, we get:

$$\underline{\underline{\Sigma}} = \frac{Nr}{2\pi V}\left[\int_{s^2} P(\underline{n}) F^c \otimes \underline{n}^c ds\right]^{\text{sym}}$$

$$= \frac{2rN}{V}\left[\underline{\underline{\Sigma}}.\underline{\underline{A}}.\frac{1}{4\pi}\int_{s^2} P(\underline{n})(\underline{n}\otimes\underline{n}) ds\right]^{\text{sym}} = \frac{2rN}{V}[\underline{\underline{\Sigma}}.\underline{\underline{A}}.\underline{\underline{D}}]^{\text{sym}}$$

for which we have used the relation:

$$\frac{1}{4\pi}\int_{s^2} P(\underline{n})(\underline{n}\otimes\underline{n}) ds = \underline{\underline{D}}\,.$$

The localization tensor $\underline{\underline{A}}$ can then be related to $\underline{\underline{D}}$ by:

$$\underline{\underline{A}}.\underline{\underline{D}} = \frac{V}{2Nr}\underline{\underline{\delta}} \Leftrightarrow \underline{\underline{A}} = \frac{V}{2Nr}\underline{\underline{D}}^{-1}\,.$$

Let us now consider again the principle of the virtual work (6). By using (23), one has:

$$\underline{\underline{\Sigma}} : \underline{\underline{E}} = \frac{1}{V}\sum_{c=1}^{N} F^c.u^c = \frac{1}{V}\sum_{c=1}^{N} \Sigma : (\underline{\underline{A}}.\underline{n}^c \otimes \underline{u}^c) \tag{24}$$

or in terms of the contact probability $P(\underline{n})$:

$$\underline{\underline{\Sigma}} : \underline{\underline{E}} = \frac{N}{4\pi V}\int_{s^2} P(\underline{n})\underline{\underline{A}}.\underline{n}^c \otimes \underline{u}^c ds : \underline{\underline{\Sigma}}\,. \tag{25}$$

As $\underline{\underline{\Sigma}}$ is symmetric*, we obtain:

$$\underline{\underline{E}} = \frac{N}{4\pi V}\left[\int_{s^2} P(\underline{n})\underline{\underline{A}}.\underline{n}^c \otimes \underline{u}^c ds\right]^{\text{sym}}\,. \tag{26}$$

Moreover, considering the inverse of the contact law (12):

$$\underline{u}^c = \underline{\underline{H}}^c.\underline{F}^c = H_n^c F_n^c \underline{n}^c + H_t^c \underline{F}_t^c \tag{27}$$

and using the static assumption (23), we obtain:

$$F_n^c = \underline{F}^c.\underline{n}^c = \underline{\underline{\Sigma}}.\underline{\underline{A}} : (\underline{n}^c \otimes \underline{n}^c)\,, \tag{28}$$

* Only the symmetrical part of the first term of the right member brings a contribution to (25).

$$
\begin{aligned}
F_t^c &= \underline{F}^c - F_n^c\,\underline{n}^c = \underline{\underline{\Sigma}}.\underline{\underline{A}}.\underline{n}^c - \underline{\underline{\Sigma}}.\underline{\underline{A}}:(\underline{n}^c \otimes \underline{n}^c \otimes \underline{n}^c) \\
&= \underline{\underline{\Sigma}}.\underline{\underline{A}}:(\underline{n}^c \otimes \underline{\underline{\delta}} - \underline{n}^c \otimes \underline{n}^c \otimes \underline{n}^c) = (\underline{\underline{\Sigma}}.\underline{\underline{A}}).\underline{n}^c.(\underline{\underline{\delta}} - \underline{n}^c \otimes \underline{n}^c).
\end{aligned} \tag{28}
$$

Now, we have to identify the overall elastic flexibility tensor of the granular medium. By the consideration of (26), (27) and of the Reuss assumption (23), we arrive at

$$
\begin{aligned}
E_{ij} = \frac{1}{2} A_{kq} \Bigg\{ & A_{ir} \frac{N}{2\pi V} \int_{s^2} P(\underline{n})\,[(H_n - H_t) n_q^c n_l^c n_j^c n_r^c + H_t n_q^c \delta_{lj} n_r^c]\, ds \\
& + A_{jr} \frac{N}{2\pi V} \int_{s} P(\underline{n})\,[(H_n - H_t) n_q^c n_l^c n_i^c n_r^c + H_t n_q^c \delta_{li} n_r^c]\, ds \Bigg\} \Sigma_{kl}.
\end{aligned} \tag{29}
$$

Since $\underline{\underline{E}} = \mathbb{S}^{\text{hom}} : \underline{\underline{\Sigma}}$, we obtain:

$$
\begin{aligned}
S_{ijkl}^{\text{hom}} = \frac{N}{2V} A_{kq} \Bigg\{ & (H_n - H_t) \left[A_{ir} \frac{1}{2\pi} \int_{s^2} P(\underline{n}) n_q n_l\, n_j n_r\, ds + A_{jk} \frac{1}{2\pi} \int_{s} P(\underline{n}) n_q n_l\, n_i\, n_r\, ds \right] \\
& + H_t \left[A_{ir} \frac{1}{2\pi} \int_{s} P(\underline{n})(n_q \delta_{lj} n_r) ds + A_{jr} \frac{1}{2\pi} \int_{s} P(\underline{n})(n_q \delta_{li} n_r) ds \right] \Bigg\}_{(kl)}.
\end{aligned} \tag{30}
$$

4.2. EXPRESSION PROVIDED BY THE USE OF THE SECOND-ORDER FABRIC TENSOR

First, we calculate the integrals

$$
\frac{1}{4\pi} \int_{s^2} P(\underline{n})\underline{n} \otimes \underline{n} \otimes \underline{n} \otimes \underline{n} ds
$$

and

$$
\frac{1}{4\pi} \int_{s^2} P(\underline{n})(\underline{n} \otimes \underline{n}) \otimes \underline{\underline{\delta}} ds
$$

intervening in (30) by using a description of the contact probability $P(\underline{n})$ based on a fabric tensor (cf. (51) and (53) in the appendix). In the same way as in the previous case of the assumption of Voigt, we demonstrate that:

$$\mathbb{S}^{\text{hom}} = \frac{N}{V}\left\{\frac{H_n - H_t}{7}\left[-\frac{1}{5}\left[\underline{\underline{A}} \otimes \underline{\underline{A}} + \underline{\underline{A}} \overline{\overline{\otimes}} \underline{\underline{A}} + \frac{1}{2}(\underline{\underline{A}}.\underline{\underline{A}}) \overline{\overline{\otimes}} \underline{\underline{\delta}} + \frac{1}{2}\underline{\underline{\delta}} \overline{\overline{\otimes}} (\underline{\underline{A}}.\underline{\underline{A}})\right]\right.\right.$$
$$+ \frac{V}{2Nr}\left[\underline{\underline{A}} \otimes \underline{\underline{\delta}} + \underline{\underline{\delta}} \otimes \underline{\underline{A}} + \frac{3}{2}(\underline{\underline{A}} \overline{\overline{\otimes}} \underline{\underline{\delta}} + \underline{\underline{\delta}} \overline{\overline{\otimes}} \underline{\underline{A}})\right]$$
$$\left.\left. + \frac{1}{2}[(\underline{\underline{A}}.\underline{\underline{A}}) \overline{\overline{\otimes}} \underline{\underline{D}} + \underline{\underline{D}} \overline{\overline{\otimes}} (\underline{\underline{A}}.\underline{\underline{A}})]\right] + H_t \frac{V}{4Nr}(\underline{\underline{A}} \overline{\overline{\otimes}} \underline{\underline{\delta}} + \underline{\underline{\delta}} \overline{\overline{\otimes}} \underline{\underline{A}})\right\}. \tag{31}$$

Remark 1: Isotropic distribution. Starting again, while the particular case of isotropic configuration of the granular medium

$$P(n) = 1 \quad \text{and} \quad \underline{\underline{D}} = \frac{1}{3}\underline{\underline{\delta}},$$

we get:

$$\mathbb{S}^{\text{hom}} = \frac{9V}{4Nr^2}\left[\frac{H_n}{15}(\underline{\underline{\delta}} \otimes \underline{\underline{\delta}} + 2\underline{\underline{\delta}} \overline{\overline{\otimes}} \underline{\underline{\delta}}) + \frac{H_t}{15}(3\underline{\underline{\delta}} \overline{\overline{\otimes}} \underline{\underline{\delta}} - \underline{\underline{\delta}} \otimes \underline{\underline{\delta}})\right].$$

The macroscopic shear and compressibility modulus are given by:

$$\mu^{\text{hom}} = \frac{10}{3}\frac{Nr^2 k_n}{V}\frac{\alpha}{3+2\alpha} \quad \text{and} \quad k^{\text{hom}} = \frac{4}{9}\frac{Nr^2 k_n}{V} \tag{32}$$

for which we recall that

$$\alpha = \frac{K_t}{K_n} = \frac{H_n}{H_t}.$$

We recover again the results provided by CHANG and LIAO [10], who have only studied the isotropic case, and associated macroscopic stiffness.

5. KINEMATICAL LOCALIZATION RULE

We aim now at proposing a general kinematic assumption relating the relative displacements in the contact $\underline{u}^c$ to the uniform deformation $\underline{\underline{E}}$ imposed on the boundary of the R.E.V. For this purpose, we take advantage of the representation theorems (see, for instance, [3]). Let us introduce the vector $\underline{U}^C$ defined by:

$$\underline{U}^c = \frac{1}{2r}P(\underline{n})\underline{u}^c. \tag{33}$$

5.1. THE KINEMATICAL LOCALIZATION RULE PROPOSED

By using the representation theorem [3], [35], we obtain:

$$\begin{aligned}\underline{U}^c &= [\hat{A}_{00} + \hat{A}_0 tr(\underline{\underline{D}}) + \hat{A}_1 tr(\underline{\underline{E}}) + \hat{A}_2 tr(\underline{\underline{E}}.\underline{\underline{D}}) + \hat{A}_3 \underline{\underline{E}} : (\underline{n}\otimes\underline{n}) + \hat{A}_4(\underline{\underline{E}}.\underline{\underline{D}}) : (\underline{n}\otimes\underline{n}) \\ &+ \hat{A}_5 tr(\underline{\underline{E}})\underline{\underline{D}} : (\underline{n}\otimes\underline{n}) + \hat{A}_6 \underline{\underline{E}} : (\underline{n}\otimes\underline{n})\underline{\underline{D}} : (\underline{n}\otimes\underline{n}) + \hat{A}_7 tr(\underline{\underline{E}})tr(\underline{\underline{D}}) + \hat{A}_8 tr(\underline{\underline{D}})(\underline{n}.\underline{\underline{E}}.\underline{n}) \\ &+ \hat{A}_9 \underline{\underline{D}} : (\underline{n}\otimes\underline{n})]\underline{n} + \underline{\underline{E}}.\underline{n}[\hat{B}_1 \underline{\underline{D}} : (\underline{n}\otimes\underline{n}) + \hat{B}_2 tr(\underline{\underline{D}}) + \hat{B}_3] + \underline{\underline{D}}.\underline{n}[\hat{C}_1 \underline{\underline{E}} : (\underline{n}\otimes\underline{n}) \\ &+ \hat{C}_2 tr\underline{\underline{E}} + \hat{C}_3] + \hat{D}_1(\underline{\underline{E}}.\underline{\underline{D}}.\underline{n} - \underline{\underline{D}}.\underline{\underline{E}}.\underline{n}) .\end{aligned} \tag{34}$$

Since $tr(\underline{\underline{D}}) = 1$, it follows that:

$$\begin{aligned}\underline{U}^c &= [A_0 + A_1 tr(\underline{\underline{E}}) + A_2 tr(\underline{\underline{E}}.\underline{\underline{D}}) + A_3 \underline{\underline{E}} : (\underline{n}\otimes\underline{n}) + A_4(\underline{\underline{E}}.\underline{\underline{D}}) : (\underline{n}\otimes\underline{n}) \\ &+ A_5 tr(\underline{\underline{E}})\underline{\underline{D}} : (\underline{n}\otimes\underline{n}) + A_6 \underline{\underline{E}} : (\underline{n}\otimes\underline{n})\underline{\underline{D}} : (\underline{n}\otimes\underline{n}) + A_7(\underline{\underline{D}}) : (\underline{n}\otimes\underline{n})]\underline{n} \\ &+ \underline{\underline{E}}.\underline{n}[B_1 \underline{\underline{D}} : (\underline{n}\otimes\underline{n}) + B_2] + \underline{\underline{D}}.\underline{n}[C_1 \underline{\underline{E}} : (\underline{n}\otimes\underline{n}) + C_2 tr\underline{\underline{E}} + C_3] + D_1(\underline{\underline{E}}.\underline{\underline{D}}.\underline{n} - \underline{\underline{D}}.\underline{\underline{E}}.\underline{n})\end{aligned} \tag{35}$$

with $A_0 = \hat{A}_0 + \hat{A}_{00}$, $A_1 = \hat{A}_1 + \hat{A}_7$, $A_3 = \hat{A}_3 + \hat{A}_8$ and $B_2 = \hat{B}_2 + \hat{B}_3$. For the other terms, we have $A_i = \hat{A}_i$, $B_i = \hat{B}_i$, $C_i = \hat{C}_i$ and $D_i = \hat{D}_i$.

Considering a linear elastic material without initial stresses and deformations (in the natural equilibrium state), in the absence of external loading without possible rigid solid motion, we obtained a linear relation between the displacement $\underline{u}^c$ and $\underline{\underline{E}}$. Therefore, for the general shape (35) of U^C one has $A_0 = A_7 = C_3 = 0$.

Now, in order to establish the general kinematic assumption linking $\underline{u}^c$ with $\underline{\underline{E}}$ (equation (33)), we start from definition (26) of the average deformation in the R.E.V.:

$$E_{ij} = \frac{1}{V}\left[\sum_{c=1}^{N}(u_i^c A_{jk} n_k^c)\right]^{\text{sym}} = \frac{N}{V}\left[\frac{1}{4\pi}\int_{s^2} P(\underline{n})(u_i^c A_{jk} n_k^c)\, ds\right]^{\text{sym}} \tag{36}$$

with $\underline{\underline{A}} = \dfrac{V}{2rN}\underline{\underline{D}}^{-1}$. Taking into account (33), it is readily seen that:

$$E_{ij} = \frac{2rN}{V}\left[\frac{1}{4\pi}\int_{s^2} U_i^c A_{jk} n_k^c\, ds\right]^{\text{sym}} . \tag{37}$$

Then, by inserting expression (35) of $\underline{U}^c$ into (37) and following the procedure detailed in the appendix B_3, it is shown that:

$$\underline{U}^c = \eta \underline{\underline{E}}.\underline{n} - \left(\frac{\eta}{2}+\frac{3}{4}\right)[5\underline{n}.\underline{\underline{E}}.\underline{n} - tr(\underline{\underline{E}})]\underline{n} + \frac{15}{2}(\underline{n}.\underline{\underline{D}}.\underline{n})\underline{\underline{E}}.\underline{n} + \beta[\underline{n} - 5\underline{\underline{D}}:(\underline{n}\otimes\underline{n})\underline{n} + 2\underline{\underline{D}}.\underline{n}]$$
$$+\lambda_1\left[-\frac{1}{35}tr(\underline{\underline{E}})\underline{n} - \frac{2}{35}tr(\underline{\underline{E}}.\underline{\underline{D}})\underline{n} + \underline{\underline{E}}:(\underline{n}\otimes\underline{n})\underline{\underline{D}}:(\underline{n}\otimes\underline{n}).\underline{n} - \frac{2}{7}\underline{\underline{D}}:(\underline{n}\otimes\underline{n})\underline{\underline{E}}.\underline{n}\right.$$
$$\left.-\frac{2}{7}\underline{\underline{E}}:(\underline{n}\otimes\underline{n})\underline{\underline{D}}.\underline{n}\right] + \lambda_2\left[-\frac{2}{5}tr(\underline{\underline{E}})\underline{n} - \frac{4}{5}tr(\underline{\underline{E}}.\underline{\underline{D}})\underline{n} + \underline{\underline{E}}:(\underline{n}\otimes\underline{n})\underline{n} - 2\underline{\underline{D}}:(\underline{n}\otimes\underline{n})\underline{\underline{E}}.\underline{n}\right.$$
$$\left.-2\underline{\underline{E}}:(\underline{n}\otimes\underline{n})\underline{\underline{D}}.\underline{n} + 4(\underline{\underline{E}}.\underline{\underline{D}}):(\underline{n}\otimes\underline{n})\underline{n} + tr(\underline{\underline{E}})\underline{\underline{D}}:(\underline{n}\otimes\underline{n})\underline{n}\right] + \lambda_3\left[-\frac{1}{2}tr(\underline{\underline{E}})\underline{n} - tr(\underline{\underline{E}}.\underline{\underline{D}})\underline{n}\right.$$
$$\left.+\frac{5}{2}\underline{\underline{E}}:(\underline{n}\otimes\underline{n})\underline{n} + 5(\underline{\underline{E}}.\underline{\underline{D}}):(\underline{n}\otimes\underline{n})\underline{n} - 5\underline{\underline{D}}:(\underline{n}\otimes\underline{n})\underline{\underline{E}}.\underline{n} + \underline{\underline{E}}.\underline{\underline{D}}.\underline{n} - \underline{\underline{D}}.\underline{\underline{E}}.\underline{n}\right]$$
$$+\lambda_4\left[-\frac{1}{2}tr(\underline{\underline{E}})\underline{n} - 2tr(\underline{\underline{E}}.\underline{\underline{D}})\underline{n} + \frac{5}{2}\underline{\underline{E}}:(\underline{n}\otimes\underline{n})\underline{n} + 10(\underline{\underline{E}}.\underline{\underline{D}}):(\underline{n}\otimes\underline{n})\underline{n} - 5\underline{\underline{D}}:(\underline{n}\otimes\underline{n})\underline{\underline{E}}.\underline{n}\right.$$
$$\left.-5\underline{\underline{E}}:(\underline{n}\otimes\underline{n})\underline{\underline{D}}.\underline{n} + tr\underline{\underline{E}}.\underline{\underline{D}}.\underline{n}\right]. \tag{38}$$

It should be noted that in the particular isotropic case, where $P(\underline{n}) = 1$ and $\underline{\underline{D}} = \frac{1}{3}\underline{\underline{\delta}}$, one has:

$$\underline{u}^c = 2r\underline{U}^c$$
$$= 2r\left\{\left[\left(A_0 + \frac{1}{3}A_7 + \frac{1}{3}C_3\right) + \left(A_1 + \frac{1}{3}A_2 + \frac{1}{3}A_5 + \frac{1}{3}C_2\right)tr(\underline{\underline{E}})\right.\right.$$
$$\left.\left.+\left(A_3 + \frac{1}{3}A_4 + \frac{1}{3}A_6 + \frac{1}{3}C_1\right)\underline{\underline{E}}:(\underline{n}\otimes\underline{n})\right]\underline{n} + \left[\frac{1}{3}B_1 + B_2\right]\underline{\underline{E}}.\underline{n}\right\} \tag{39}$$

which leads to:

$$\underline{u}^c = 2r\left\{\left[\left(\beta - \frac{5}{3}\beta + \frac{2}{3}\beta\right) + \left(\frac{3}{4} + \frac{1}{2}\eta - \frac{1}{21}\lambda_1 - \frac{1}{3}\lambda_2 - \frac{5}{6}\lambda_3 - \frac{5}{6}\lambda_4\right)tr(\underline{\underline{E}})\right.\right.$$
$$\left.+\left(-\frac{15}{4} - \frac{5}{2}\eta - \frac{5}{21}\lambda_1 - \frac{5}{3}\lambda_2 - \frac{25}{6}\lambda_3 - \frac{25}{6}\lambda_4\right)\underline{\underline{E}}:(\underline{n}\otimes\underline{n})\right]\underline{n}$$
$$\left.+\left[\eta + \frac{5}{2} - \frac{2}{21}\lambda_1 + \frac{5}{3}\lambda_2 - \frac{5}{3}\lambda_3 - \frac{5}{3}\lambda_4\right]\underline{\underline{E}}.\underline{n}\right\}. \tag{40}$$

Then, we obtain the following kinematical assumption:

$$\underline{u}^c = 2r\left\{\left(\eta + \frac{5}{2} - \frac{2}{21}\lambda_1 - \frac{2}{3}\lambda_2 - \frac{5}{3}\lambda_3 - \frac{5}{3}\lambda_4\right)\underline{\underline{E}}.\underline{n}\right.$$
$$\left. - \left(\frac{3}{4} + \frac{\eta}{2} - \frac{1}{21}\lambda_1 - \frac{1}{3}\lambda_2 - \frac{5}{6}\lambda_3 - \frac{5}{6}\lambda_4\right)[5\underline{n}.\,\underline{\underline{E}}.\underline{n} - tr(\underline{\underline{E}})]\underline{n}\right\}. \quad (41)$$

It appears that the expression of $\underline{u}^c$ as the function of $\underline{\underline{E}}$ is not affected by *β*. On the other hand, it is interesting to notice that in this isotropic case, for the particular value

$$\eta - \frac{2}{21}\lambda_1 - \frac{2}{3}\lambda_2 - \frac{5}{3}\lambda_3 - \frac{5}{3}\lambda_4 = -\frac{3}{2},$$

we recover the kinematic Voigt assumption (see equation (4)):

$$\underline{U}^c = 2r\underline{\underline{E}}.\underline{n}\,.$$

5.2. DETERMINATION OF THE MACROSCOPIC ELASTIC STIFFNESS TENSOR

The anisotropic kinematical rule proposed here is obviously more general than that of Voigt. It depends on five adjustable parameters λ_1, λ_2, λ_3, λ_4, β as well as on $\underline{\underline{D}}$. We limit here the study to the particular case of one adjustable parameter* η by considering $\lambda_i = 0$, $i = 1$ to 4 and $\beta = 0$. This choice allows us to reduce the number of the free parameters without the loss of generality regarding the anisotropy description:

$$\underline{u}^c = \frac{2r}{P(\underline{n})}\left\{\eta\underline{\underline{E}}.\underline{n} - \left(\frac{\eta}{2} + \frac{3}{4}\right)[5\underline{n}.\underline{\underline{E}}.\underline{n} - tr(\underline{\underline{E}})]\,\underline{n} + \frac{15}{2}\underline{\underline{D}} : (\underline{n}\otimes\underline{n})\underline{\underline{E}}.\underline{n}\right\}. \quad (42)$$

We aim now at applying the homogenization scheme based on this kinematic assumption (42). In the same way as the assumption of Voigt, we consider together the expression of the average stress tensor (8), the contact law (11) and the kinematical rule (42) to determine the macroscopic properties of the granular medium.

To that end let us decompose first the displacement in the local basis associated with a given contact. We have:

$$\underline{u}_n^c = \underline{u}^c.\underline{n} = \frac{2r}{P(\underline{n})}\left\{-\left(\frac{3\eta}{2} + \frac{15}{4}\right)\underline{\underline{E}} : (\underline{n}\otimes\underline{n}) + \left(\frac{\eta}{2} + \frac{3}{4}\right)tr(\underline{\underline{E}}) + \frac{15}{2}\underline{\underline{D}} : (\underline{n}\otimes\underline{n})\underline{\underline{E}} : (\underline{n}\otimes\underline{n})\right\},$$

---

* The general study of the kinematic assumption (38) is to do again in order to specify the physical significance of the parameters λ_i. However, it does not constitute the object of the work presented here.

$$\underline{u}_t^c = (\underline{u}^c - \underline{u}_n^c \underline{n}) = \frac{2r}{P(\underline{n})}\left\{\left[\eta + \frac{15}{2}\underline{\underline{D}} : (\underline{n} \otimes \underline{n})\right]\underline{\underline{\underline{T}}} : \underline{\underline{E}}\right\},$$

where $\underline{\underline{\underline{T}}} = \underline{\underline{\underline{I}}}.\underline{n} - \underline{n} \otimes \underline{n} \otimes \underline{n}$ and $\underline{\underline{\underline{T}}} : \underline{\underline{E}} = \underline{\underline{E}}.\underline{n} - \underline{\underline{E}} : (\underline{n} \otimes \underline{n} \otimes \underline{n})$ due to the symmetry of $\underline{\underline{E}}$. On the other hand, with the linear elastic contact law, we have:

$$\underline{F}_n^c = K_n \underline{u}_n^c = \frac{2r}{P(\underline{n})}\left\{K_n\left[-\left(\frac{3\eta}{2} + \frac{15}{4}\right)\underline{\underline{E}} : (\underline{n} \otimes \underline{n}) + \left(\frac{\eta}{2} + \frac{3}{4}\right)tr(\underline{\underline{E}}) + \frac{15}{2}\underline{\underline{D}} : (\underline{n} \otimes \underline{n})\right]\underline{\underline{E}} : (\underline{n} \otimes \underline{n})\right\}$$

and

$$\underline{F}_t^c = K_t \underline{u}_t^c = \frac{2r}{P(\underline{n})}\left\{\left[\eta + \frac{15}{2}\underline{\underline{D}} : (\underline{n} \otimes \underline{n})\right]K_t \underline{\underline{\underline{T}}} : \underline{\underline{E}}\right\}.$$

Now, inserting these two last expressions into (8), we arrive at:

$$\underline{\underline{\Sigma}} = \frac{4Nr^2}{V}\frac{1}{4\pi}\int_{s^2}\left\{K_n\left[-\left(\frac{3\eta}{2} + \frac{15}{4}\right)\underline{\underline{E}} : (\underline{n} \otimes \underline{n}) + \frac{15}{2}\underline{\underline{D}} : (\underline{n} \otimes \underline{n} \otimes \underline{n} \otimes \underline{n}) : \underline{\underline{E}} + \left(\frac{\eta}{2} + \frac{3}{4}\right)tr\underline{\underline{E}}\right](\underline{n} \otimes \underline{n}) + \left[\eta + \frac{15}{2}\underline{\underline{D}} : (\underline{n} \otimes \underline{n})\right]K_t \underline{\underline{\underline{T}}}^T . \underline{\underline{\underline{T}}} : \underline{\underline{E}}\right\}ds$$

with $(\underline{F}_t^c \otimes \underline{n})^{\mathrm{sym}} = \underline{\underline{\underline{T}}}^T . \underline{F}_t^c$ which reads also:

$$\underline{\underline{\Sigma}} = \frac{4r^2N}{V}\left\{K_n\left[-\left(\frac{3\eta}{2} + \frac{15}{4}\right)\frac{1}{4\pi}\int_{s^2}\underline{n} \otimes \underline{n} \otimes \underline{n} \otimes \underline{n}ds + \left(\frac{\eta}{2} + \frac{3}{4}\right)\frac{1}{4\pi}\int_{s^2}\underline{n} \otimes \underline{n} \otimes \underline{\underline{\delta}}ds + \frac{15}{2}\underline{\underline{D}} : \frac{1}{4\pi}\int_{s^2}\underline{n} \otimes \underline{n} \otimes \underline{n} \otimes \underline{n} \otimes \underline{n} \otimes \underline{n}ds\right] + K_t\left[\eta\frac{1}{4\pi}\int_{s^2}\underline{\underline{\underline{T}}}^T . \underline{\underline{\underline{T}}}ds + \frac{15}{2}\underline{\underline{D}} : \frac{1}{4\pi}\int_{s^2}(\underline{n} \otimes \underline{n}) \otimes \underline{\underline{\underline{T}}}^T . \underline{\underline{\underline{T}}}ds\right]\right\} : \underline{\underline{E}}. \tag{43}$$

By identification with $\underline{\underline{\Sigma}} = \mathbb{C}^{\mathrm{hom}} : \underline{\underline{E}}$, we obtain the following expression for the homogenized elastic stiffness tensor (cf. (54), (55), (56), (57) and (58) in the appendix B_2):

$$\mathbb{C}^{\text{hom}} = \frac{4r^2N}{V} K_n \left\{ -\left(\frac{\eta}{2} + \frac{5}{4}\right)\left[\frac{1}{5}(\underline{\underline{\delta}} \otimes \underline{\underline{\delta}}) + \frac{2}{5}(\underline{\underline{\delta}} \mathbin{\overline{\otimes}} \underline{\underline{\delta}})\right] + \left(\frac{\eta}{6} + \frac{1}{4}\right)\underline{\underline{\delta}} \otimes \underline{\underline{\delta}} \right.$$

$$\left. + \left[\frac{1}{14}(\underline{\underline{\delta}} \otimes \underline{\underline{\delta}} + 2\underline{\underline{\delta}} \mathbin{\overline{\otimes}} \underline{\underline{\delta}}) + \frac{1}{7}(\underline{\underline{D}} \otimes \underline{\underline{\delta}} + \underline{\underline{\delta}} \otimes \underline{\underline{D}} + 2\underline{\underline{D}} \mathbin{\overline{\otimes}} \underline{\underline{\delta}} + 2\underline{\underline{\delta}} \mathbin{\overline{\otimes}} \underline{\underline{D}}\right]\right\}$$

$$+ \frac{4r^2N}{V} K_t \left\{ -\frac{\eta}{3}\left[\frac{1}{5}(\underline{\underline{\delta}} \otimes \underline{\underline{\delta}}) + \frac{2}{5}(\underline{\underline{\delta}} \mathbin{\overline{\otimes}} \underline{\underline{\delta}})\right]\right.$$

$$\left. -\left[\frac{1}{14}(\underline{\underline{\delta}} \otimes \underline{\underline{\delta}} - 5\underline{\underline{\delta}} \mathbin{\overline{\otimes}} \underline{\underline{\delta}}) + \frac{1}{7}\left(\underline{\underline{D}} \otimes \underline{\underline{\delta}} + \underline{\underline{\delta}} \otimes \underline{\underline{D}} - \frac{3}{2}\underline{\underline{D}} \mathbin{\overline{\otimes}} \underline{\underline{\delta}} - \frac{3}{2}\underline{\underline{\delta}} \mathbin{\overline{\otimes}} \underline{\underline{D}}\right)\right] + \frac{\eta}{3}(\underline{\underline{\delta}} \mathbin{\overline{\otimes}} \underline{\underline{\delta}}) \right\}. \quad (44)$$

Note again that for $\eta = -3/2$, we recover the macroscopic stiffness tensor provided by the Voigt assumption.

5.3. THE CASE OF AN ISOTROPIC DISTRIBUTION OF CONTACTS

In the isotropic case, it is convenient to compare the results given by the kinematical rule with, on the one hand, those already obtained from the Voigt assumption and the Reuss assumption and, on the other hand, with the existing numerical results obtained by CHANG and LIAO [10]. In the isotropic case, expression (44) becomes*:

$$\mathbb{C}^{\text{hom}} = \frac{4Nr^2}{V}\left\{ K_n \left[\left(\frac{\eta}{15} + \frac{1}{6}\right)\underline{\underline{\delta}} \otimes \underline{\underline{\delta}} - \left(\frac{\eta}{5} + \frac{1}{6}\right)\underline{\underline{\delta}} \mathbin{\overline{\otimes}} \underline{\underline{\delta}}\right]\right.$$

$$\left. + K_t \left[-\left(\frac{\eta}{15} + \frac{1}{6}\right)\underline{\underline{\delta}} \otimes \underline{\underline{\delta}} + \left(\frac{\eta}{5} + \frac{1}{2}\right)\underline{\underline{\delta}} \mathbin{\overline{\otimes}} \underline{\underline{\delta}}\right]\right\}$$

from which it follows that:

$$\mu^{\text{hom}} = \frac{E}{2(1+\nu)} = \frac{1}{15}\frac{Nr^2k_n}{V}(-(5+6\eta) + 3\alpha(5+2\eta)),$$
$$k^{\text{hom}} = \frac{E}{3(1-2\nu)} = \frac{4}{9}\frac{Nr^2k_n}{V}. \quad (45)$$

It must be noticed that:

• for $\eta = -\frac{3}{2}$ the results agree, as already indicated, with those obtained from the Voigt assumption,

---

* We recall that in the isotropic case, the utilization of a fabric tensor of the second or fourth order leads to the same result.

• for $\eta = -\frac{5}{2}\left(\frac{1+2\alpha}{3+2\alpha}\right)$ the results agree with those obtained by the Reuss assumption.

6. ANALYSIS OF THE RESULTS

The results obtained from the three localization rules (Voigt, Reuss and kinematical) for the isotropic case are presented in the table.

Comparison of the localizations considered

	The Voigt assumption	The Reuss assumption	The kinematical rule proposed
k^{hom}	$\frac{4}{9}\frac{Nr^2K_n}{V}$	$\frac{4}{9}\frac{Nr^2K_n}{V}$	$\frac{4}{9}\frac{Nr^2K_n}{V}$
μ^{hom}	$\frac{2}{15}\frac{Nr^2K_n}{V}(2+3\alpha)$	$\frac{10}{3}\frac{Nr^2K_n}{V}\frac{\alpha}{3+2\alpha}$	$\frac{1}{15}\frac{Nr^2K_n}{V}(-(5+6\eta)+3\alpha(5+2\eta))$

Based on the results in the table it can be concluded that the macroscopic compressibility k^{hom} is independent of the localization assumption used. It depends only in a linear way on the normal rigidity in the contact K_n.

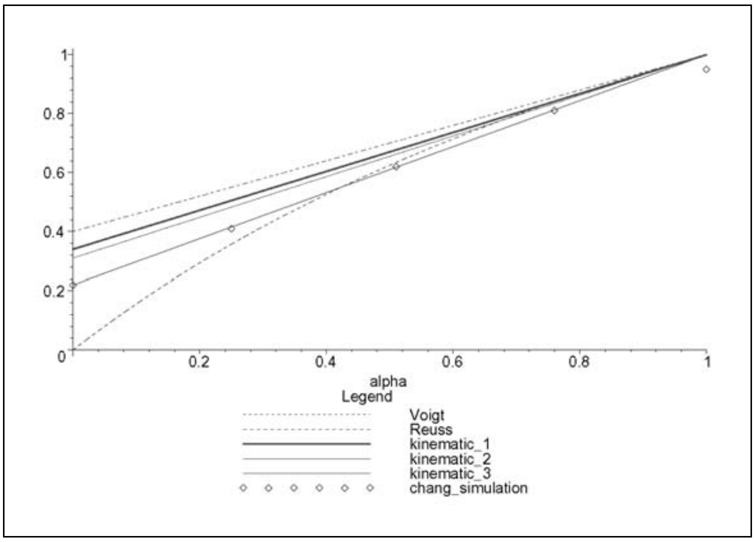

Fig. 4. Variation of $\frac{3V}{2Nr^2k_n}\mu^{\text{hom}}$ with respect to α by using the assumptions of Voigt, Reuss, kinematic 1 ($\eta = -1.4$), kinematic 2 ($\eta = -1.35$) and kinematic 3 ($\eta = -1.2$)

We also observe (cf. figure (4)) that the variation of μ^{hom} obtained from the kinematical rule is linear with respect to α. On the other hand, when α tends to 1, the curve of μ^{hom} has a tendency to be out the limit observation given by the Reuss assumption. It must be recalled that the Reuss results do not provide a lower bound. This is in agreement with the numerical results reported in [10], which correspond to the discrete points added in figure 4. For $\eta = -1.2$, the variations of μ^{hom} (linear in α) seems to be in agreement with the numerical results (see figure 5).

Thus, the general kinematic assumption proposed here allows us to approach accurately the numerical simulation results.

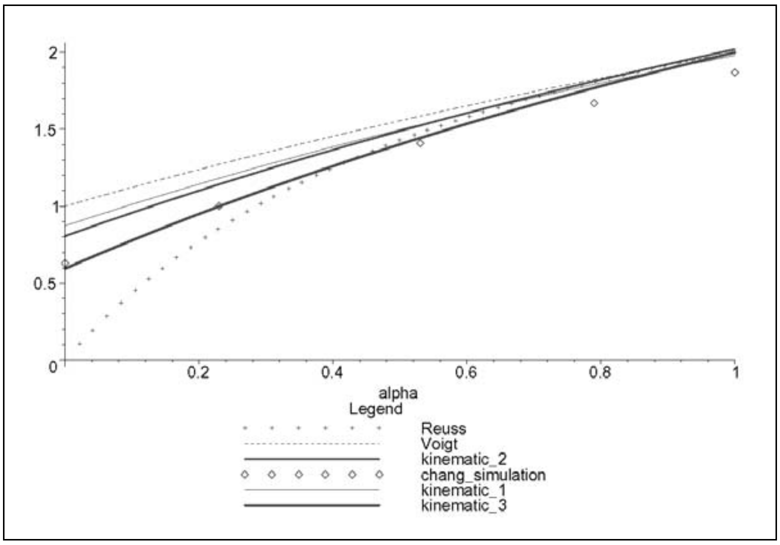

Fig. 5. Variation of $\dfrac{3V}{2Nr^2k_n}E^{\mathrm{hom}}$ with respect to α by using the assumptions of Voigt, Reuss, kinematic 1 ($\eta = -1.4$), kinematic 2 ($\eta = -1.35$) and kinematic 3 ($\eta = -1.2$)

7. CONCLUSION

In this paper, we proposed an upscaling approach which aims to describe the anisotropic elastic properties of the granular materials. This approach takes advantage of the representation of the grain contacts' distribution by means of the second-order fabric tensor. It appears particularly relevant when the materials exhibit orthotropic symmetry. The consideration of this fabric tensor in various homogenization schemes allows us to quantify the overall anisotropy of the material.

In particular, we obtain new results based on the kinematical localization rule proposed. In the case of an isotropic material, it is shown that the predictions of elastic moduli deduced from the new kinematical rule are in a very good agreement with the numerical results of CHANG and LIAO [10]. This agreement provides the first validation of the approach developed. Obviously, for anisotropic granular media, it will be interesting to proceed with a more complete validation, for instance, by performing numerical discrete computations of elastic moduli.

REFERENCES

[1] BATHURST R.J., ROTHENBERG L., *Micromechanical aspects of isotropic granular assemblies with linear contact interactions*, Journal of Applied Mechanics ASME, 1988, Vol. 55, 17–23.

[2] BATHURST R.J., ROTHENBURG L., *Observations on stress force-fabric relationships in idealized granular materials*, Mech. Mater., 1990, No. 9, 65–80.

[3] BOEHLER J.P., *Application of tensors functions in solids mechanics*, CISM courses and lectures, No. 292, Springer-Verlag, Wien, New York, 1987.

[4] CAMBOU B., DUBUJET Ph., EMERIAULT F., SIDOROFF F., *Homogenization for granular materials*, European Journal of Mechanics, A/Solids, 1995, Vol. 14, No. 2, 225–276.

[5] CAMBOU B., CHAZE M., DEDECKER F., *Change of scale in granular materials*, European Journal of Mechanics, A/Solids, 2000, No. 19, 999–1014.

[6] CAMBOU B., JEAN M., *Micromécanique des materiaux granulaires*, Hermes Science, 2001.

[7] CAMBOU B., DUBUJET PH., NOUGUIER-LEHON C., *Anisotropy in granular materials at different scales*, Mechanics of materials, 2004, 1–10.

[8] CHANG C.S., *Micromechanical modeling of constitutive equation for granular materials*, Micromechanics of granular materials, Elsevier Science Publishers, 1988, 271–278.

[9] CHANG C.S., LIAO C.L., *Constitutive relations for particulate medium with the effect of particle rotations*, International Journal of Solids and Structures, 1990, Vol. 26, No. 4, 437–453.

[10] CHANG C.S., LIAO C.L., *Estimates of elastic moduli for media of randomly packed granules*, Appl. Mech. Rev., 1994, Vol. 47, No. 1, Part 2, 197–207.

[11] CHANG C.S., CHAO S.J., CHANG Y., *Estimates of elastic moduli for granular material with anisotropic random packing structure*, International Journal of Solids and Structures, 1995, Vol. 32, No. 14, 1989–2008.

[12] CHANG C.S., GAO J., *Kinematic and static hypotheses for constitutive modelling of granulates considering particle rotation*, Acta Mech., 1996, No. 115, 213–229.

[13] CHAPUIS R.B., *De la structure des milieux granulaires en relation avec leur comportement mécanique*, PhD Thesis, Montréal, Canada, 1976.

[14] CHRISTOFFERSEN J., MEHRABADI M.M., NEMAT-NASSER S., *A micromechanical description of granular material behavior*, Journal of Applied Mechanics, ASME, 1981, Vol. 48, No. 2, 339–344.

[15] EMERIAULT F., CAMBOU B., MAHBOUBI A., *Homogenization for granular materials: non reversible behaviour*, Mechanics of Cohesive-Frictional Materials, 1996, Vol. 1, 199–218.

[16] EMERIAULT F., CHANG C.S., *Interparticle forces and displacement in granular materials*, Computer and Geotechnics, 1997, 20 (3/4), 223–244.

[17] FORTIN J., *Simulation numérique de la dynamique des systèmes multicorps appliquée aux milieux granulaires*, PhD Thesis, University of Lille, 2000.

[18] HE Q.-C., CURNIER A., *A more fundamental approach to damaged elastic stress–strain relations*, Int. J. Solids Structures, 1995, Vol. 32, No. 10, 1433–1457.

[19] JENKINS J.T., *Anisotropic elasticity for random arrays of identical spheres*, Modern Theory of Anisotropic Elasticity and Applications, J. Wu. (Ed)., SIAM, Philadelphia, 1991.

[20] JENKINS J.T., *Inelastic behavior of random arrays of identical spheres*, Fleck N.A. and Cocks A.C.E. (Eds), IUTAM Symposium on Mechanics of Granular and Porous Materials, Kluwer Academic Publishers, 1997, 11–22.

[21] KANATANI K., *Distribution of directional data and fabric tensors*, Int. J. Engng Sci., 1984, Vol. 22, No. 2, 149–164.

[22] KRAJCINOVIC D., *Damage mechanics*, North-Holland, The Netherlands, 1996.

[23] LIAO C.-L., CHANG T.-C., YOUNG C.S., *Stress–strain relationship for granular materials based on the hypothesis of the best fit*, Int. J. Solids. Struct., 1997, Vol. 34, 4010–4087.

[24] LOUIS L., DAVID C., METZ V., ROBION P., MENENDZ B., KISSEL C., *Microstructural control on the anisotropy and transport properties in undeformed sandstones*, International Journal of Rock Mechanics and Mining Sciences, 2005, No. 42, 911–923.

[25] LOUIS L., ROBION P., DAVID C., FRIZON DE LAMOTTE D., *Multiscale anisotropy controlled by folding: the example of the Chaudrons fold (Corbieres, France)*, J. Struct. Geology, 2006, Vol. 28, No. 4, 549–560.

[26] LOVE A.E.H., *A treatise of mathematical theory of elasticity*, University Press, Cambridge, 1927.

[27] LUBARDA V.A., KRAJCINOVIC D., *Damage tensors and the crack density distribution*, Int. J. Solids Structures, 1993, Vol. 30, No. 20, 2859–2877.

[28] MADADI M., TSOUNGUI O., LÄTZEL M., LUDING S., *On the fabric tensor of polydisperse granular materials in 2D*, International Journal of Solids and Structures, 2004, No. 41, 2563–2580.

[29] NOUGIER C., *Simulation des interactions outil-sol : application aux outils de traitement des sols*, Thèse de doctorat du Département de Modélisation Physique et Numérique de l'Institut de Physique du Globe de Paris, 1999.

[30] NOUGUIER-LEHON C., DUBUJET P., CAMBOU B., *Analysis of granular material behaviour from two kinds of numerical modelling*, 15th ASCE Engineering Mechanics Conference, Columbia University, New York, NY, June 2002.

[31] PENSEE V., *Contribution de la micromécanique à la modélisation tridimensionnelle de l'endommagement par mésofissuration*, PhD Thesis, University of Lille, 2002.

[32] ROTHENBURG L., SELVADURAI A.P.S., *Micromechanical definition of the Cauchy stress tensor for particulate media*, Mechanics of Structured Media, Elsevier, Amsterdam, The Netherlands, 1981, 469–486.

[33] SATAKE M., *Constitution of mechanics of granular materials through graph theory*, Cowin S.C. et Satake M. (Eds.), Proc. US-Japan Seminar on Continuum Mechanical and Statistical Approaches in the Mechanics of Granular Materials, Elsevier, Amsterdam, The Netherlands, 1978, 47–62.

[34] SIDOROFF F., CAMBOU B., MAHBOUBI A., *Contact force distribution in granular media*, Mechanics of Materials, 1993, No. 16, 83–89.

[35] SPENCER A.J.M., *Isotropic polynomial invariants and tensor functions*, J.P. Beohler (ed.), Applications of Tensor Functions in Solid Mechanics, CISM Courses and Lectures No. 292, Springer, Berlin, 1987, 141–169.

[36] WALTON K., *The effective elastic moduli of random packing of spheres*, Journal of Mechanics and Physics of Solids, 1987, Vol. 35, No. 3, 213–226.

[37] WEBER J., *Recherches concernant les contraintes intergranulaires dans les milieux pulvirulents*, bul. liaison P. et Ch., No. juil.–août 1966.

APPENDICES

A. TENSORIAL NOTATIONS

We are going to use in what follows the results of HE and CURNIER [18]:

$$\frac{1}{4\pi}\int_{s^2} ds = 1,$$

$$\frac{1}{4\pi}\int_{s^2} n_i n_j ds = \frac{1}{3}\delta_{ij},$$

$$\frac{1}{4\pi}\int_{s^2} n_i n_j n_k n_l ds = \frac{1}{5}J_{ijkl}, \tag{46}$$

$$\frac{1}{4\pi}\int_{s^2} n_i n_j n_k n_l n_m n_n ds = \frac{1}{7}J_{ijklmn},$$

$$\frac{1}{4\pi}\int_{s^2} n_i n_j n_k n_l n_m n_n n_p n_q ds = \frac{1}{9}J_{ijklmnpq}$$

with:

$$J_{ijkl} = \frac{1}{3}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

$$J_{ijklmn} = \frac{1}{3}(\delta_{ij}J_{klmn} + \delta_{ik}J_{jlmn} + \delta_{il}J_{jkmn} + \delta_{im}J_{jkln} + \delta_{in}J_{jklm}),$$

$$J_{ijklmnpq} = \frac{1}{7}(\delta_{ij}J_{klmnpq} + \delta_{ik}J_{jlmnpq} + \delta_{il}J_{jkmnpq} + \delta_{im}J_{jklnpq} + \delta_{in}J_{jklmpq} + \delta_{ip}J_{jklmnq} + \delta_{iq}J_{jklmnp}).$$

B. APPROXIMATION WITH A SECOND-ORDER TENSOR

B.1. EXPRESSION OF THE CONTACT PROBABILITY

According to LUBARDA and KRAJCINOVIC [27], we approximate the distribution of the contact probability $P(\underline{n})$ in the normal direction $\underline{n}$ by a second-order tensor $\underline{\underline{d}}$ as $P(\underline{n}) = \underline{\underline{d}} : (\underline{n} \otimes \underline{n})$. It can be shown that $P(\underline{n})$ can be expressed with respect to the macroscopic variable $\underline{\underline{D}}$. Indeed, in 3D, we have:

$$\underline{\underline{D}} = \frac{1}{4\pi}\int_{s^2} P(\underline{n})(\underline{n}\otimes\underline{n})ds \quad \text{and} \quad tr\underline{\underline{D}} = 1 \tag{47}$$

from which one gets:

$$\underline{\underline{d}} = \frac{15}{2}\left[\underline{\underline{D}} - \frac{1}{5}(tr\underline{\underline{D}})\underline{\underline{\delta}}\right] = \frac{15}{2}\left[\underline{\underline{D}} - \frac{1}{5}\underline{\underline{\delta}}\right]. \tag{48}$$

Indeed:

$$\underline{\underline{D}} = \frac{1}{4\pi}\int_{s^2} P(\underline{n})(\underline{n}\otimes\underline{n})ds = \underline{\underline{d}} : \frac{1}{4\pi}\int_{s^2} \underline{n}\otimes\underline{n}\otimes\underline{n}\otimes\underline{n}ds$$
$$= \underline{\underline{d}} : \frac{1}{15}(\underline{\underline{\delta}}\otimes\underline{\underline{\delta}} + 2\underline{\underline{\delta}}\,\overline{\underline{\otimes}}\,\underline{\underline{\delta}}) = \frac{1}{15}[tr(\underline{\underline{d}})\underline{\underline{\delta}} + 2\underline{\underline{d}}].$$

We have then $tr(\underline{\underline{D}}) = \frac{1}{3}tr(\underline{\underline{d}}) = 1$, and $tr(\underline{\underline{d}}) = 3$. From these results, $\underline{\underline{d}}$ is given by $\underline{\underline{d}} = \frac{15}{2}\left[(\underline{\underline{D}}) - \frac{1}{5}\underline{\underline{\delta}}\right]$.

It follows that:

$$P(\underline{n}) = \frac{15}{2}\left[\underline{\underline{D}} : (\underline{n}\otimes\underline{n}) - \frac{1}{5}\right]. \tag{49}$$

B.2. COMPUTATION OF SOME REQUIRED INTEGRALS

We use expression (49) of $P(\underline{n})$ and the results (46) of HE and CURNIER [18] to compute the following integral:

$$\frac{1}{4\pi}\int_{s^2} P(\underline{n})(\underline{n}\otimes\underline{n})ds = \frac{1}{4\pi}\int_{s^2}\frac{15}{2}\left[\underline{\underline{D}} : (\underline{n}\otimes\underline{n}) - \frac{1}{5}\right](\underline{n}\otimes\underline{n})ds$$
$$= \frac{1}{4\pi}\left[\frac{15}{2}\underline{\underline{D}} : \int_{s^2} \underline{n}\otimes\underline{n}\otimes\underline{n}\otimes\underline{n}ds - \frac{3}{2}\int_{s^2}(\underline{n}\otimes\underline{n})ds\right] = \underline{\underline{D}}. \tag{50}$$

In the same way, from (49) and (46), the following identity can be established:

$$\frac{1}{4\pi}\int_{s^2} P(\underline{n})(\underline{n}\otimes\underline{n}\otimes\underline{n}\otimes\underline{n})ds = \frac{1}{4\pi}\int_{s^2}\frac{15}{2}\left[\underline{\underline{D}} : (\underline{n}\otimes\underline{n}) - \frac{1}{5}\right](\underline{n}\otimes\underline{n}\otimes\underline{n}\otimes\underline{n})ds$$
$$= \frac{1}{4\pi}\left[\frac{15}{2}\underline{\underline{D}} : \int_{s^2} (\underline{n}\otimes\underline{n}\otimes\underline{n}\otimes\underline{n}\otimes\underline{n}\otimes\underline{n})ds - \frac{3}{2}\int_{s^2}(\underline{n}\otimes\underline{n}\otimes\underline{n}\otimes\underline{n})ds\right]$$
$$= \frac{1}{7}\left[-\frac{1}{5}(\underline{\underline{\delta}}\otimes\underline{\underline{\delta}}) - \frac{2}{5}(\underline{\underline{\delta}}\,\overline{\underline{\otimes}}\,\underline{\underline{\delta}}) + (\underline{\underline{D}}\otimes\underline{\underline{\delta}} + \underline{\underline{\delta}}\otimes\underline{\underline{D}}) + 2(\underline{\underline{D}}\,\overline{\underline{\otimes}}\,\underline{\underline{\delta}} + \underline{\underline{\delta}}\,\overline{\underline{\otimes}}\,\underline{\underline{D}})\right]. \tag{51}$$

Knowing that $\underline{\underline{\underline{T}}} = \underline{n}.\underline{\underline{\underline{\underline{I}}}} - \underline{n}\otimes\underline{n}\otimes\underline{n}$ and $\underline{\underline{\underline{T}}}^T = \underline{\underline{\underline{\underline{I}}}}.\underline{n} - \underline{n}\otimes\underline{n}\otimes\underline{n}$, we have:

$$\frac{1}{4\pi}\int_{s^2} P(\underline{n})(\underline{\underline{\underline{T}}}^T.\underline{\underline{\underline{T}}})ds = \frac{15}{2}D : \frac{1}{4\pi}\int_{s^2} (\underline{n}\otimes\underline{n})\otimes[\underline{\underline{\delta}}\,\overline{\underline{\otimes}}\,(\underline{n}\otimes\underline{n}) + (\underline{n}\otimes\underline{n}\,\overline{\underline{\otimes}})\underline{\underline{\delta}}]ds$$
$$-\frac{3}{4}\frac{1}{4\pi}\int_{s^2}[\underline{\underline{\delta}}\,\overline{\underline{\otimes}}\,(\underline{n}\otimes\underline{n}) + (\underline{n}\otimes\underline{n})\,\overline{\underline{\otimes}}\,\underline{\underline{\delta}}]ds - \frac{1}{4\pi}\int_{s^2} P(\underline{n})\otimes(\underline{n}\otimes\underline{n}\otimes\underline{n}\otimes\underline{n})ds$$
$$= \frac{1}{7}\left[\frac{1}{5}\underline{\underline{\delta}}\otimes\underline{\underline{\delta}} + \frac{2}{5}(\underline{\underline{\delta}}\,\overline{\underline{\otimes}}\,\underline{\underline{\delta}}) - (\underline{\underline{D}}\otimes\underline{\underline{\delta}} + \underline{\underline{\delta}}\otimes\underline{\underline{D}}) + \frac{3}{2}(\underline{\underline{D}}\,\overline{\underline{\otimes}}\,\underline{\underline{\delta}} + \underline{\underline{\delta}}\,\overline{\underline{\otimes}}\,\underline{\underline{D}})\right]. \tag{52}$$

In the same way, we compute the integral:

$$\frac{1}{4\pi}\int_{s^2} P(\underline{n})\underline{n}\otimes\underline{\underline{\delta}}\otimes\underline{n}\,ds = \frac{1}{4\pi}\int_{s^2}\left[\frac{15}{2}\underline{\underline{D}}:\left(\underline{n}\otimes\underline{n}-\frac{3}{2}\right)\right]\underline{n}\otimes\underline{\underline{\delta}}\otimes\underline{n}\,ds$$

$$=\frac{15}{2}\underline{\underline{D}}:\frac{1}{4\pi}\int_{s^2}\underline{n}\otimes\underline{n}\otimes\underline{n}\otimes\underline{\underline{\delta}}\otimes\underline{n}\,ds-\frac{3}{2}\frac{1}{4\pi}\int_{s^2}\underline{n}\otimes\underline{\underline{\delta}}\otimes\underline{n}\,ds=\underline{\underline{D}}\otimes\underline{\underline{\delta}}\,. \tag{53}$$

The following identities are also obtained by using equations (46):

$$\frac{1}{4\pi}\int_{s^2}\underline{n}\otimes\underline{n}\otimes\underline{n}\otimes\underline{n}\,ds=\frac{1}{15}(\underline{\underline{\delta}}\otimes\underline{\underline{\delta}}+2\underline{\underline{\delta}}\,\overline{\otimes}\,\underline{\underline{\delta}}) \tag{54}$$

and

$$\frac{1}{4\pi}\int_{s^2}\underline{n}\otimes\underline{n}ds\otimes\underline{\underline{\delta}}=\frac{1}{3}\underline{\underline{\delta}}\otimes\underline{\underline{\delta}}\cdot \tag{55}$$

In the same way, by using (46), we have:

$$\underline{\underline{D}}:\frac{1}{4\pi}\int_{s^2}\underline{n}\otimes\underline{n}\otimes\underline{n}\otimes\underline{n}\otimes\underline{n}\otimes\underline{n}ds=\frac{1}{105}(\underline{\underline{\delta}}\otimes\underline{\underline{\delta}}+2\underline{\underline{\delta}}\overline{\otimes}\underline{\underline{\delta}})$$

$$+\frac{2}{105}(\underline{\underline{D}}\otimes\underline{\underline{\delta}}+\underline{\underline{\delta}}\otimes\underline{\underline{D}}+2\underline{\underline{D}}\,\overline{\otimes}\,\underline{\underline{\delta}}+2\underline{\underline{\delta}}\,\overline{\otimes}\,\underline{\underline{D}}) \tag{56}$$

and

$$\frac{1}{4\pi}\int_{s^2}\underline{\underline{T}}^T.\underline{\underline{T}}ds=\frac{1}{4\pi}\int_{s^2}\left\{\frac{1}{2}[(\underline{n}\otimes\underline{n})\,\overline{\otimes}\,\underline{\underline{\delta}}+\underline{\underline{\delta}}\overline{\otimes}(\underline{n}\otimes\underline{n})]-\underline{n}\otimes\underline{n}\otimes\underline{n}\otimes\underline{n}\right\}ds$$

$$=\frac{1}{3}\underline{\underline{\delta}}\,\overline{\otimes}\,\underline{\underline{\delta}}-\frac{1}{15}(\underline{\underline{\delta}}\otimes\underline{\underline{\delta}}+2\underline{\underline{\delta}}\,\overline{\otimes}\,\underline{\underline{\delta}})\,. \tag{57}$$

Finally, we have:

$$\underline{\underline{D}}:\frac{1}{4\pi}\int_{s^2}(\underline{n}\otimes\underline{n})\otimes\underline{\underline{T}}^T.\underline{\underline{T}}ds$$

$$=\underline{\underline{D}}:\frac{1}{4\pi}\int_{s^2}\left\{\frac{1}{2}[(\underline{n}\otimes\underline{n}\otimes\underline{n}\otimes\underline{n})\overline{\otimes}\underline{\underline{\delta}}+\underline{n}\otimes\underline{n}\otimes\underline{\underline{\delta}}\overline{\otimes}(\underline{n}\otimes\underline{n})]-\underline{n}\otimes\underline{n}\otimes\underline{n}\otimes\underline{n}\otimes\underline{n}\otimes\underline{n}\right\}ds$$

$$=\frac{1}{15}(\underline{\underline{\delta}}\overline{\otimes}\underline{\underline{\delta}}+\underline{\underline{D}}\overline{\otimes}\underline{\underline{\delta}}+\underline{\underline{\delta}}\overline{\otimes}\underline{\underline{D}})-\left[\frac{1}{105}(\underline{\underline{\delta}}\otimes\underline{\underline{\delta}}+2\underline{\underline{\delta}}\overline{\otimes}\underline{\underline{\delta}})+\frac{2}{105}(\underline{\underline{D}}\otimes\underline{\underline{\delta}}+\underline{\underline{\delta}}\otimes\underline{\underline{D}}+2\underline{\underline{D}}\overline{\otimes}\underline{\underline{\delta}}+2\underline{\underline{\delta}}\overline{\otimes}\underline{\underline{D}})\right], \tag{58}$$

where use has been made of the following definition:

$$(\mathbb{A}\,\overline{\otimes}\,\underline{\underline{B}})_{ijklmn}=\frac{1}{2}(A_{ijkn}B_{lm}+A_{ijknm}B_{ln})\cdot$$

B.3. ELEMENTS OF THE KINEMATICAL LOCALIZATION RULE

$$E_{ij} = \frac{2Nr}{V}\Bigg\{A_{kj}\Bigg[A_0 \frac{1}{4\pi}\int_{s^2} n_i n_k ds + A_1 tr(\underline{\underline{E}})\frac{1}{4\pi}\int_{s^2} n_i n_k ds + A_2 tr(\underline{\underline{E}}.\underline{\underline{D}})\frac{1}{4\pi}\int_{s^2} n_i n_k ds$$
$$+ A_3 E_{pq}\frac{1}{4\pi}\int_{s^2} n_p n_q n_i n_k ds + A_4(\underline{\underline{E}}.\underline{\underline{D}})_{pq}\frac{1}{4\pi}\int_{s^2} n_p n_q n_i n_k ds$$
$$+ A_5 tr(\underline{\underline{E}}) D_{pq}\frac{1}{4\pi}\int_{s^2} n_p n_q n_i n_k ds + A_6 E_{pq}\frac{1}{4\pi}\int_{s^2} n_p n_q n_i n_k n_m n_n ds D_{nm}$$
$$+ A_7 D_{pq}\frac{1}{4\pi}\int_{s^2} n_p n_q n_i n_k ds + B_1 E_{in}\frac{1}{4\pi}\int_{s^2} n_n n_q n_p n_k ds D_{pq}$$
$$+ B_2 E_{ip}\frac{1}{4\pi}\int_{s^2} n_p n_k ds + C_1 D_{in}\frac{1}{4\pi}\int_{s^2} n_n n_q n_p n_k ds E_{pq} + C_2 tr(\underline{\underline{E}}) D_{ip}\frac{1}{4\pi}\int_{s^2} n_p n_k ds$$
$$+ C_3 D_{ip}\frac{1}{4\pi}\int_{s^2} n_p n_k ds + D_1(\underline{\underline{E}}.\underline{\underline{D}} - \underline{\underline{D}}.\underline{\underline{E}})_{ip}\int_{s^2} n_p n_k ds\Bigg]\Bigg\}^{\text{sym}}. \tag{59}$$

Knowing that $\underline{\underline{A}}.\underline{\underline{D}} = \frac{V}{2Nr}\underline{\underline{\delta}}$, and after having calculated the integrals of (59) (cf. the appendix B_2), we get:

$$E_{ij} = \frac{2Nr}{V}\Bigg\{A_0 A_{ij} + A_1 tr(\underline{\underline{E}})A_{ij} + A_2 tr(\underline{\underline{E}}.\underline{\underline{D}})A_{ij} + \frac{1}{5}A_3[tr(\underline{\underline{E}})A_{ij} + 2E_{ik}A_{kj}]$$
$$+\frac{1}{5}A_4\left[tr(\underline{\underline{E}}.\underline{\underline{D}})A_{ij} + \frac{V}{2Nr}E_{ij} + E_{kl}D_{li}A_{kj}\right] + \frac{1}{5}A_5\left[tr(\underline{\underline{E}})A_{ij} + \frac{V}{Nr}tr(\underline{\underline{E}})\delta_{ij}\right]$$
$$+\frac{1}{35}A_6\left[tr(\underline{\underline{E}})A_{ij} + \frac{V}{Nr}tr(\underline{\underline{E}})\delta_{ij} + 2tr(\underline{\underline{E}}.\underline{\underline{D}})A_{ij} + 4E_{lk}A_{kj}D_{il} + 2\frac{V}{Nr}E_{ij} + 2E_{ik}A_{kj}\right]$$
$$+\frac{1}{5}A_7\left[A_{ij} + \frac{V}{Nr}\delta_{ij}\right] + \frac{1}{5}B_1\left[\frac{V}{Nr}E_{ij} + E_{ik}A_{kj}\right] + B_2 E_{ik}A_{kj}$$
$$+\frac{1}{5}C_1\left[2E_{kl}D_{li}A_{kj} + \frac{V}{2Nr}tr(\underline{\underline{E}})\delta_{ij}\right] + C_2\frac{V}{2Nr}tr(\underline{\underline{E}})\delta_{ij} + C_3\frac{V}{2Nr}\delta_{ij} + D_1\left[\frac{V}{2Nr}E_{ij} - E_{lk}A_{kj}D_{il}\right]\Bigg\}^{\text{sym}} \tag{60}$$

which after symmetrization gives:

$$E_{ij} = \frac{2Nr}{3V}\Bigg\{A_0 A_{ij} + A_1 tr(\underline{\underline{E}})A_{ij} + A_2 tr(\underline{\underline{E}}.\underline{\underline{D}})A_{ij} + \frac{1}{5}A_3[tr(\underline{\underline{E}})A_{ij} + E_{ik}A_{kj} + E_{jk}A_{ki}]$$
$$+\frac{1}{5}A_4\left[tr(\underline{\underline{E}}.\underline{\underline{D}})A_{ij} + \frac{V}{2Nr}E_{ij} + \frac{1}{2}E_{kl}D_{li}A_{kj} + \frac{1}{2}E_{kl}D_{lij}A_{ki}\right] + \frac{1}{5}A_5\left[tr(\underline{\underline{E}})A_{ij} + \frac{V}{Nr}tr(\underline{\underline{E}})\delta_{ij}\right]$$
$$+\frac{1}{35}A_6\left[\frac{V}{Nr}tr(\underline{\underline{E}})\delta_{ij} + tr(\underline{\underline{E}}.\underline{\underline{D}})A_{ij} + 2E_{lk}A_{kj}D_{il} + 2E_{lk}A_{ki}D_{jl} + \frac{V}{Nr}E_{ij} + E_{ik}A_{kj} + E_{jk}A_{ki}\right]$$
$$+\frac{1}{5}A_7\left[A_{ij} + \frac{V}{Nr}\delta_{ij}\right] + \frac{1}{5}B_1\left[\frac{V}{Nr}E_{ij} + \frac{1}{2}E_{ik}A_{kj} + \frac{1}{2}E_{jk}A_{ki}\right] + \frac{1}{2}B_2[E_{ik}A_{kj} + E_{jk}A_{ki}]$$
$$+\frac{1}{5}C_1\left[E_{kl}D_{li}A_{kj} + E_{kl}D_{lj}A_{ki} + \frac{V}{2Nr}tr(\underline{\underline{E}})\delta_{ij}\right] + C_2\frac{V}{2Nr}tr(\underline{\underline{E}})\delta_{ij} + C_3\frac{V}{2Nr}\delta_{ij} - \frac{1}{2}D_1[E_{lk}A_{kj}D_{il} + E_{lk}A_{ki}D_{jl}]\Bigg\}^{\text{sym}}. \tag{61}$$

Since this last relation must be satisfied for any value of $\underline{\underline{D}}$ and $\underline{\underline{E}}$, we arrive at the following system of 8 equations with 14 unknowns, i.e. A_0 to A_7, B_1, B_2, C_1, C_2, C_3 and D_1:

$$
\begin{cases}
A_0 + \frac{1}{5}A_7 = 0, \quad \frac{1}{5}A_7 + \frac{1}{2}C_3 = 0,\\
A_1 + \frac{1}{5}A_3 + \frac{1}{5}A_5 + \frac{1}{35}A_6 = 0,\\
A_2 + \frac{1}{5}A_4 + \frac{2}{35}A_6 = 0,\\
\frac{1}{5}A_3 + \frac{1}{35}A_6 + \frac{1}{10}B_1 + \frac{1}{2}B_2 = 0,\\
\frac{1}{5}A_5 + \frac{1}{35}A_6 + \frac{1}{10}C_1 + \frac{1}{2}C_2 = 0,\\
\frac{1}{10}A_4 + \frac{2}{35}A_6 + \frac{1}{5}C_1 - \frac{1}{2}D_1 = 0,\\
\frac{1}{15}A_4 + \frac{4}{105}A_6 + \frac{2}{15}B_1 + \frac{1}{3}D_1 = 1.
\end{cases}
$$

To solve this system, let us fix 6 of the 14 unknowns, for example, $A_6 = \lambda_1$, $A_5 = \lambda_2$, $D_1 = \lambda_3$, $C_2 = \lambda_4$, $A_0 = \beta$, $B_2 = \eta$, and let us express the other unknowns with respect to λ_i, β and η. We have:

$$
\begin{cases}
A_0 = \beta, \quad B_2 = \eta,\\
A_1 = \frac{3}{4} + \frac{\eta}{2} = \frac{1}{35}\lambda_1 - \frac{2}{5}\lambda_2 - \frac{1}{2}\lambda_3 - \frac{1}{2}\lambda_4,\\
A_2 = -\left(\frac{2}{35}\lambda_1 + \frac{4}{5}\lambda_2 + \lambda_3 + 2\lambda_4\right),\\
A_3 = -\frac{15}{4} - \frac{5}{2}\eta + \lambda_2 + \frac{5}{2}\lambda_3 + \frac{5}{2}\lambda_4,\\
A_4 = 4\lambda_2 + 5\lambda_3 + 10\lambda_4,\\
A_5 = \lambda_2, \quad A_6 = \lambda_1, \quad A_7 = -5\beta,\\
B_1 = \frac{15}{2} - \frac{2}{7}\lambda_1 - 2\lambda_2 - 5\lambda_3 - 5\lambda_4,\\
C_1 = -\frac{2}{7}\lambda_1 - 2\lambda_2 - 5\lambda_4,\\
C_2 = \lambda_4, \quad C_3 = 2\beta, \quad D_1 = \lambda_3.
\end{cases}
$$

The parameters λ_1, λ_2, λ_3, λ_4, β and η are the constants which can be considered as the characteristics of the internal state of the granular medium. Then, by introducing the values of A_i, B_i, C_i and D_i in expression (35) of U^c, we obtain the general kinematic assumption given by (38).

B.4. COMPONENTS OF THE HOMOGENIZED STIFFNESS IN THE VOIGT NOTATION

By adopting for $\mathbb{C}^{\text{hom}}$ the classic notation of Voigt which allows $\mathbb{C}^{\text{hom}}$ to be represented by a symmetrical matrix 6×6, we obtain from (20):

$$C_{11}=\left[6D_{11}-\frac{3}{5}\right]K_n+\left[D_{11}+\frac{3}{5}\right]K_t\,,\qquad C_{12}=\left[D_{11}+D_{22}-\frac{1}{5}\right](K_n-K_t)$$

$$C_{13}=\left[D_{11}+D_{33}-\frac{1}{5}\right](K_n-K_t)\,,\qquad C_{14}=D_{23}(K_n-K_t)\,,$$

$$C_{15}=D_{13}\left(3K_n+\frac{1}{2}K_t\right),\qquad C_{16}=D_{12}\left(3K_n+\frac{1}{2}K_t\right),$$

$$C_{22}=\left[6D_{22}-\frac{3}{5}\right]K_n+\left[D_{22}+\frac{3}{5}\right]K_t\,,\qquad C_{23}=\left[D_{22}+D_{33}-\frac{1}{5}\right](K_n-K_t)\,,$$

$$C_{24}=D_{23}\left(3K_n+\frac{1}{2}K_t\right),\qquad C_{25}=D_{31}(K_n-K_t)\,,$$

$$C_{26}=D_{12}\left(3K_n+\frac{1}{2}K_t\right),\qquad C_{33}=\left[6D_{33}-\frac{3}{5}\right]K_n+\left[D_{33}+\frac{3}{5}\right]K_t\,,$$

$$C_{34}=D_{23}\left(3K_n+\frac{1}{2}K_t\right),\qquad C_{35}=D_{31}\left(3K_n+\frac{1}{2}K_t\right),$$

$$C_{44}=\left[D_{22}+D_{33}-\frac{1}{5}\right]K_n+\left[\frac{3}{4}D_{33}+\frac{3}{4}D_{22}+\frac{1}{5}\right]K_t\,,\qquad C_{36}=D_{12}(K_n-K_t)\,,$$

$$C_{45}=D_{12}\left(K_n+\frac{3}{4}K_t\right),\qquad C_{46}=D_{13}\left(K_n+\frac{3}{4}K_t\right),$$

$$C_{55}=\left[D_{11}+D_{33}-\frac{1}{5}\right]K_n+\left[\frac{3}{4}D_{11}+\frac{3}{4}D_{33}+\frac{1}{5}\right]K_t\,,\qquad C_{56}=D_{32}\left(K_n+\frac{3}{4}K_t\right),$$

$$C_{66}=\left[D_{11}+D_{22}-\frac{1}{5}\right]K_n+\left[\frac{3}{4}D_{22}+\frac{3}{4}D_{11}+\frac{1}{5}\right]K_t\,.$$