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EQUATIONS OF BIOT'S CONSOLIDATION WITH KELVIN–VOIGHT RHEOLOGICAL FRAME

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Abstract: Taking thermodynamics of irreversible processes as the starting point, the constitutive compounds of the Biot–Darcy consolidation model were derived, provided that the porous medium's skeleton is the Kelvin–Voight viscoelastic body. The physical compounds obtained make it possible to formulate the equations of Biot's consolidation process within a rheological framework.

Streszczenie: Wychodząc z termodynamiki procesów nieodwracalnych dla przypadku procesu izotermicznego, wyprowadzono związki konstytutywne modelu konsolidacji Biota–Darcy'ego, gdy szkielet ośrodka porowatego jest ciałem lepkosprężystym Kelvina–Voighta. Uzyskane związki fizyczne pozwalają sformułować równania procesu konsolidacji Biota ze szkieletem reologicznym.

Резюме: Исходя из термодинамики неотвратимых процессов для случая изотермического процесса, были выведены конститутивные связи модели консолидации Биота–Дарси, когда скелет пористой среды является вязко-упругим телом Кельвина–Войта. Полученые физические связи дают возможность формулировки уравнений процесса консолидации Биота с реологическим скелетом.

1. INITIAL ASSUMPTIONS FOR MODEL CONSTRUCTION

The starting point of the survey is the theory of a biphasic medium consisting of a solid phase, called the medium skeleton, and a fluid phase (composed of liquid and gas) flowing through the skeleton pores. The model analysed is based on the Biot–Darcy poroelastic theory, in accordance with the works by ARRIAULT [1]–[3], BIOT [5], [6] and COUSSY [7]. The model in question, however, additionally takes into account viscous properties of a saturated soil skeleton.

The assumptions of the mathematical model of consolidation are as follows:

- The medium is biphasic. It consists of a viscoelastic porous skeleton and a low-compressible Newtonian fluid filling the skeleton pores.
- The solid phase is the Kelvin–Voight body.
- The skeleton deformations are small; it is possible, therefore, to omit the non-linear terms of the strain tensor.
- The stresses in the porous medium skeleton are related to the total surface of horizontal section (averaged stress).

• As far as the fluid is concerned, the term of the pore fluid stress σ has been introduced, linked with the effective fluid pressure by the relation expressed as $\sigma = -f \cdot p$, where f stands for the volumetric porosity of the porous medium, and p for the effective fluid pressure.

• The porosity f of a medium is considered a constant value of statistical importance.

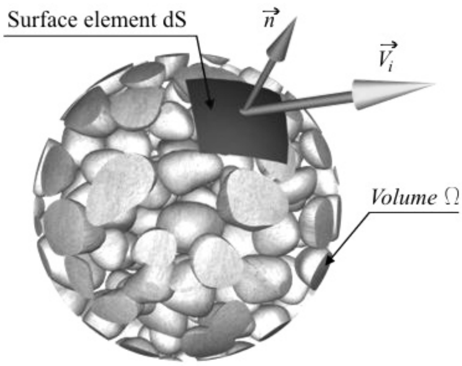
• In the description of processes, the Lagrange system of reference was employed.

2. EQUATION OF BIPHASIC MEDIUM FLOW CONTINUITY

Assuming that Ω describes an elementary region filled with a biphasic medium, let us take S to stand for the limiting surface through which the fluid filtration flow takes place. $\vec{n}$ denotes a versor normal to S, directed towards the outside of the region (the figure), and a total mass flow (of the skeleton and the fluid) through the wall with the surface area S equals:

$$\int_S \rho v_i^s n_i dS + \int_S \overline{\rho}(v_i^l - v_i^s) n_i dS + \int_\Omega \frac{\partial \rho}{\partial t} d\Omega = 0\,, \tag{1}$$

where: $\vec{v}^l$ and $\vec{v}^s$ – the vectors of the velocity of the fluid flow and the displacement of the medium skeleton, respectively, ρ – the biphasic medium density, $\overline{\rho}$ – the fluid density.



Elementary computational volume

Using the Gauss–Ostrogradski theorem, the equation of flow continuity (according to the work by STRZELECKI et al. [9]) of a biphasic medium, consisting of the fluid and the skeleton, takes the form of a local compound:

$$\frac{D^s \rho}{Dt} + \rho \dot{\varepsilon} = -[\overline{\rho}\, v_i^r]_i , \tag{2}$$

where $\frac{D^s}{Dt}$ denotes the material derivative expressed by the following formula:

$$\frac{D^s}{Dt} = \frac{\partial}{\partial t} + v_i^s \frac{\partial}{\partial x_i},$$

whereas $\dot{\varepsilon}$ stands for the velocity of the change of skeleton dilatation, equal, as far as the value is concerned, to $v_{i,i}^s$. Analogically, for the fluid phase of the medium, the fluid flow through the surface S – according to STRZELECKI et al. [9] – can be represented by:

$$\int_S \overline{\rho}\, v_i^s n_i dS + \int_S \overline{\rho}(v_i^l - v_i^s) n_i dS + \int_\Omega \frac{\partial \overline{\rho}}{\partial t} d\Omega = 0 . \tag{3}$$

Hence, the equation of the continuity of the fluid phase flow is defined by:

$$\frac{D^r \overline{\rho}}{Dt} + \overline{\rho}(\dot{\theta} - \dot{\varepsilon}) = -(\overline{\rho}\, v_i^l)_{,i} , \tag{4}$$

where $\frac{D^r}{Dt}$ is a material derivative expressed by the formula:

$$\frac{D^r}{Dt} = \frac{\partial}{\partial t} + (v_i^l - v_i^s) \frac{\partial}{\partial x_i} .$$

Assuming that the solid phase is immobile $(v_i^S = 0)$, whereas the compressible fluid infiltrates through the pores, the equation of flow continuity has *raison d'être* only with reference to the fluid phase of the medium and is equivalent to:

$$\mathrm{div}(\overline{\rho}\, \vec{v}) = -\frac{\partial(\overline{\rho})}{\partial t} . \tag{5}$$

Such a form of the equation of the continuity of a non-deformable medium will be obtained for the hydrodynamic model of a filtration flow (STRZELECKI [9]).

3. EQUATIONS OF SOLID AND FLUID PHASE MOTION

The equations of the solid and the fluid phase motion were described in detail by BIOT [5]. Momentum conservation law of the solid phase of the medium may be expressed by the formula:

$$\int_S \sigma_{ij} n_i dS + \int_\Omega b(v_i^l - v_i^s) d\Omega + \int_\Omega (\rho - \overline{\rho}) X_i d\Omega = \int_\Omega \frac{D^s P_i^s}{Dt} d\Omega\,, \tag{6}$$

where $\sigma_{ij} n_j$ stands for the stresses in the skeleton interacting with the surface S, whereas X_i are the gravity forces exerted on a total mass unit.

Using the Gauss–Ostrogradski theorem, equation (6) assumes the form of a local equation of the solid phase motion of the medium:

$$\sigma_{ij,j} + X_i(\rho - \overline{\rho}) = -b v_i^r + \rho_{11} \frac{D^S v_i^S}{Dt} + \rho_{12} \frac{D^S v_i^l}{Dt}\,, \tag{7}$$

where $\frac{D^S}{Dt}$ is a material derivative expressed by the formula:

$$\frac{D^s}{Dt} = \frac{\partial}{\partial t} + v_i^s \frac{\partial}{\partial x_i}\,.$$

For the fluid phase of the medium, the momentum conservation law is equivalent to:

$$\int_S \sigma\, n_i ds + \int_\Omega b(v_i^s - v_i^l) d\Omega + \int_\Omega \overline{\rho} X_i d\Omega = \int_\Omega \frac{D^l P_i^l}{dt} d\Omega\,, \tag{8}$$

where $\frac{D^l}{Dt}$ is a material derivative represented by the formula:

$$\frac{D^l}{Dt} = \frac{\partial}{\partial t} + v_i^l \frac{\partial}{\partial x_i}\,;$$

σn_i stands for the stresses in the fluid related to the total surface S.

Once the Gauss–Ostrogradski theorem is used, the momentum conservation law makes it possible to determine the equations of the fluid phase motion of the medium as follows:

$$\sigma_{,i} + X_i \overline{\rho} = b v_i^r + \rho_{12} \frac{D^l v_i^s}{Dt} + \rho_{12} \frac{D^l v_i^l}{Dt}\,. \tag{9}$$

As far as the quasi-static flow is concerned, it is possible to neglect the terms that represent the inertial forces of the fluid and the skeleton. In that case, the equations for each of the phases may be given by:

$$\begin{gathered} \sigma_{ij,j} + X_i(\rho - \overline{\rho}) = -b v_i^r\,, \\ \sigma_{,i} + X_i \overline{\rho} = b v_i^r\,. \end{gathered}\,, \tag{10}$$

4. DETERMINATION OF CONSTITUTIVE COMPOUNDS FOR ISOTHERMAL PROCESS

Let us consider the constitutive compounds for the Biot–Darcy consolidation process, taking into account the case of the skeleton of the Kelvin–Voight porous medium. We will obtain constitutive equations by departing from the first law of thermodynamics of irreversible processes and using the methodology presented by de GROOT and MAZUR [8]. The first law of thermodynamics may be represented by the following equation:

$$\dot{L} + \dot{Q} = \frac{D}{Dt}(W + K)\,, \tag{11}$$

where: L is the algebraic sum of work done by external forces, body forces resulting from the influence of gravitational field, and body forces generated by the influence of the resistance of a filtering motion and by the passage of electric current; $\dot{Q}$ stands for the variation of heat with time; W signifies the internal energy of the medium, and K denotes kinetic energy.

Following the methodology described by BARTLEWSKA [4], we will write the first law of thermodynamics of irreversible processes for each of the phases separately, using indicator 1 for the solid phase of the medium and indicator 2 – for the fluid phase.

The power of the internal forces of skeleton is expressed by:

$$\dot{L}_1^A = \int_S (\sigma_{ij} + \sigma\delta_{ij}) v_i^s n_j dS\,, \tag{12}$$

where: $\sigma_{ij} = \sigma_{ij}^s + \sigma_{ij}^{\mathrm{lep}}$, and σ_{ij}^s is the tensor of the elastic stress in the skeleton, depending on the strain tensor ε_{ij}, $\sigma_{ij}^{\mathrm{lep}}$ is the tensor of viscous stress in the skeleton, depending on the deformation velocity $\dot{\varepsilon}_{ij}$.

The power of the terrestrial gravity forces of the medium skeleton amounts to:

$$\dot{L}_1^P = \int_\Omega (\rho - \overline{\rho}) X_i v_i^s d\Omega\,. \tag{13}$$

The power of the viscous resistance forces of the fluid, related to the skeleton, is equal to:

$$\dot{L}_1^D = -\int_\Omega b v_i^r v_i^s d\Omega\,. \tag{14}$$

Since the power is a scalar quantity, the total power of the forces interacting with the medium skeleton equals:

$$\dot{L}_1 = \dot{L}_1^A + \dot{L}_1^P + \dot{L}_1^D\,. \tag{15}$$

The velocity of heat variation in the medium skeleton is expressed by:

$$\dot{Q}_1 = -\int_{\Omega} q_{i,i}^s d\Omega , \tag{16}$$

where q_i^s are the components of the heat flux flowing through the solid phase of the medium.

The material derivative of kinetic energy for the solid phase of the medium amounts to:

$$\frac{D^S K_1}{Dt} = \int_{\Omega} \left(\rho_{11} v_i^s \frac{D^S v_i^s}{Dt} + \rho_{12} v_i^s \frac{D^S v_i^l}{Dt} \right) d\Omega . \tag{17}$$

The material derivative of internal energy for the skeleton equals:

$$\frac{D^S W}{Dt} = \int_{\Omega} \dot{w}_1 d\Omega , \tag{18}$$

where $\dot{w}_1$ stands for the velocity of the local internal energy of the skeleton.

Bearing in mind the formulae from (13) to (18), the first law of thermodynamics with reference to the solid phase of the medium may be represented by:

$$\int_{\Omega} \left(\dot{w}_1 + \rho_{11} v_i^s \frac{D^s v_i^s}{Dt} + \rho_{12} v_i^s \frac{D^s v_i^l}{Dt} \right) d\Omega$$
$$= \int_{\Omega} [X_i (\rho - \overline{\rho}) v_i^s - b v_i^r v_i^s + (\sigma_{ij} + \sigma \delta_{ij}) \dot{\varepsilon}_{ij} + (\sigma_{ij.j} + \sigma_{,j} \delta_{ij}) v_i^s - q_{i,i}^s] d\Omega. \tag{19}$$

Having employed the fluid motion equation, equation (19) may be shown as a local compound which expresses the value of free energy in the solid phase of the medium in the following way:

$$\dot{w}_1 = \sigma_{,i} v_i^s + (\sigma_{ij} + \sigma \delta_{ij}) \dot{\varepsilon}_{ij} - q_{i,j}^s . \tag{20}$$

For the fluid, the power of internal forces equals:

$$\dot{L}_2^A = \int_S \sigma v_i^r n_i dS . \tag{21}$$

The power of the terrestrial gravity forces of the fluid can be defined as:

$$\dot{L}_2^P = \int_{\Omega} \overline{\rho} X_i v_i^l d\Omega , \tag{22}$$

and the power of viscous resistance in the fluid:

$$\dot{L}_2^D = \int_\Omega (b_{11} v_i^r + b_{12} j_i) v_i^l d\Omega \,. \tag{23}$$

The velocity of heat variation in the fluid amounts to:

$$\dot{Q} = -\int_\Omega q_{i,i}^l d\Omega \,, \tag{24}$$

where q_i^l are the components of the heat flux vector.

The material derivative of the kinetic energy K_2 of the fluid is equal to:

$$\frac{D^k K_2}{Dt} = \int_\Omega \left(\rho_{12} v_i^l \frac{D^l v_i^s}{Dt} + \rho_{12} v_i^l \frac{D^l v_i^l}{Dt} \right) d\Omega \,. \tag{25}$$

The material derivative of the internal energy of the fluid in the area of Ω may be expressed by the formula:

$$\frac{D^l W_2}{Dt} = \int_\Omega \dot{w}_2 d\Omega \,. \tag{26}$$

Employing formulae (21)–(26), the first law of thermodynamics for the fluid is expressed by the following relation:

$$\int_\Omega \left(\dot{w}_2 + \rho_{22} v_i^l \frac{D^l v_i^l}{Dt} + \rho_{12} v_i^l \frac{D^l v_i^s}{Dt} \right) d\Omega$$

$$= \int_\Omega [X_i \bar{\rho} v_i^l - b v_i^r v_i^l + \sigma(\dot{\theta} - \dot{\varepsilon}) + \sigma_{,i}(v_i^l - v_i^s) - q_{i,i}^l] d\Omega, \tag{27}$$

where $\dot{\theta}$ denotes the dilatation velocity of the fluid, and $\dot{\varepsilon}$ stands for the dilatation velocity of the medium skeleton. Taking into account the equation of fluid motion, equation (9) may be written in the form of a local compound:

$$\dot{w}_2 = -\sigma_{,i} v_i^s + \sigma(\dot{\theta} - \dot{\varepsilon}) - q_{i,i}^l \,. \tag{28}$$

It may be assumed that the velocity $\dot{w}$ of the internal energy variation of the bi-phasic medium is equal to the algebraic sum of the velocities $\dot{w}_1$ and $\dot{w}_2$ of each of energy variations of the medium phases, therefore:

$$\dot{w} = \dot{w}_1 + \dot{w}_2 \,. \tag{29}$$

If q_i denotes the components of the velocity of the heat flow of the biphasic medium (the skeleton + the fluid), then it may be stated that:

$$q_i = q_i^s + q_i^l . \tag{30}$$

Using relations (29) and (30), as well as equations (20) and (28), it is possible to calculate the variation of the internal energy of the biphasic medium:

$$\dot{w} = \sigma_{ij}\dot{\varepsilon}_{ij} + \sigma\dot{\theta} - q_{i,i} . \tag{31}$$

In order to obtain constitutive compounds, we will employ the Helmholtz free energy definition given by:

$$F = W - ST. \tag{32}$$

The Helmholtz free energy, just as the internal energy W, is the function of the medium state. If the alterations of the medium state are infinitely small, then the thermodynamic function F (function of state) may be expressed by means of the definition of a total differential dF:

$$dF = dW - TdS - SdT . \tag{33}$$

In the case of isothermal processes, we have:

$$dT = 0 . \tag{34}$$

Therefore, in that case:

$$dF = dW - TdS . \tag{35}$$

If the total differential of the Helmoltz function F is treated as the change of that function of state with time, then equation (35) may be formulated in the following way:

$$\dot{F} = \dot{W} - T\dot{S} = \dot{W} - T[\dot{S}_w + \dot{S}_z] . \tag{36}$$

We also introduce a local function of the Helmholtz free energy χ which satisfies the equation:

$$F = \int_{\Omega} \chi \, d\Omega .$$

On the local scale, equation (36) assumes the following form:

$$\dot{\chi} = \sigma\dot{\varepsilon} + \sigma\dot{\theta} - T\dot{s}_w .$$

Hence, we obtain:

$$T\dot{s}_w = \sigma_{ij}\dot{\varepsilon}_{ij} + \sigma\dot{\theta} - \dot{\chi} \geq 0 . \tag{37}$$

Substituting:

$$\chi = \chi(\varepsilon_{ij}, \dot{\varepsilon}_{ij}, \theta)\,, \tag{38}$$

on the basis of inequality (37), we arrive at:

$$\left(\sigma_{ij}^{s} - \frac{\partial \chi}{\partial \varepsilon_{ij}}\right)\dot{\varepsilon}_{ij} + \left(\sigma_{ij}^{\mathrm{lep}} - \frac{\partial \chi}{\partial \varepsilon_{ij}}\right)\dot{\varepsilon}_{ij} + \left(\sigma - \frac{\partial \chi}{\partial \theta}\right)\dot{\theta} \geq 0\,, \tag{39}$$

which gives, as a result:

$$\left(\sigma_{ij}^{s} - \frac{\partial \chi}{\partial \varepsilon_{ij}} - \frac{\partial \chi}{\partial \dot{\varepsilon}_{ij}}\right)\dot{\varepsilon}_{ij} + \left(\sigma - \frac{\partial \chi}{\partial \theta}\right)\dot{\theta} \geq 0\,. \tag{40}$$

Equation (40) is satisfied for each $\dot{\varepsilon}_{ij}$ and $\dot{\theta}$, provided that

$$\sigma_{ij} = \frac{\partial \chi}{\partial \varepsilon_{ij}} + \frac{\partial \chi}{\partial \dot{\varepsilon}_{ij}} \tag{41}$$

and

$$\sigma = \frac{\partial \chi}{\partial \theta}\,. \tag{42}$$

Since the function of the variation of the Helmholtz free energy is a total differential, we have:

$$d\chi = \frac{\partial \chi}{\partial \varepsilon_{ij}}\dot{\varepsilon}_{ij} + \frac{\partial \chi}{\partial \dot{\varepsilon}_{ij}}\dot{\varepsilon}_{ij} + \frac{\partial \chi}{\partial \theta}\dot{\theta}\,. \tag{43}$$

Expanding the free energy function χ into the Taylor series around the natural state leads to:

$$\begin{aligned}
\chi(\varepsilon_{ij}, \dot{\varepsilon}_{ij}, \theta) &= \chi(0,0) + \frac{\partial \chi(0,0)}{\partial \varepsilon_{ij}}\varepsilon_{ij} + \frac{\partial \chi(0,0)}{\partial \dot{\varepsilon}_{ij}}\dot{\varepsilon}_{ij} + \frac{\partial \chi(0,0)}{\partial \theta}\theta \\
&+ \frac{1}{2}\left[\frac{\partial^2 \chi(0,0)}{\partial \varepsilon_{ij}\partial \varepsilon_{kl}}\varepsilon_{ij}\varepsilon_{kl} + \frac{\partial^2 \chi(0,0)}{\partial \dot{\varepsilon}_{ij}\partial \dot{\varepsilon}_{kl}}\dot{\varepsilon}_{ij}\dot{\varepsilon}_{kl} + \frac{\partial^2 \chi(0,0)}{\partial^2 \theta}\theta\theta + 2\frac{\partial^2 \chi(0,0)}{\partial \varepsilon_{ij}\partial \dot{\varepsilon}_{kl}}\varepsilon_{ij}\dot{\varepsilon}_{kl}\right. \\
&\left. + 2\frac{\partial^2 \chi(0,0)}{\partial \varepsilon_{ij}\partial \theta}\varepsilon_{ij}\theta + 2\frac{\partial^2 \chi(0,0)}{\partial \dot{\varepsilon}_{ij}\partial \theta}\dot{\varepsilon}_{ij}\theta\right] + \ldots
\end{aligned} \tag{44}$$

In the natural (non-deformed) state, the functions: $\chi(0,0)$, $\sigma_{ij}(0,0)$ and $\sigma(0,0)$ are equal to zero, therefore, with the accuracy to small values of the second order, we may write:

$$2\chi(\varepsilon_{ij},\theta) = c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \eta_{ijkl}\dot{\varepsilon}_{ij}\dot{\varepsilon}_{kl} + \gamma\theta\theta + d_{ijkl}\varepsilon_{ij}\dot{\varepsilon}_{kl} + \lambda_{ij}\dot{\varepsilon}_{ij}\theta + \beta_{ij}\varepsilon_{ij}\theta\,, \tag{45}$$

where:

$$c_{ijkl} = \frac{\partial^2\chi(0,0)}{\partial\varepsilon_{ij}\partial\varepsilon_{kl}}, \quad \eta_{ijkl} = \frac{\partial^2\chi(0,0)}{\partial\dot{\varepsilon}_{ij}\partial\dot{\varepsilon}_{kl}}, \quad \gamma = \frac{\partial^2\chi(0,0)}{\partial\theta\,\partial\theta},$$

$$d_{ijkl} = 2\frac{\partial^2\chi(0,0)}{\partial\varepsilon_{ij}\partial\dot{\varepsilon}_{kl}}, \quad \lambda_{ij} = 2\frac{\partial^2\chi(0,0)}{\partial\dot{\varepsilon}_{ij}\partial\theta}, \quad \beta_{ij} = 2\frac{\partial^2\chi(0,0)}{\partial\varepsilon_{ij}\partial\theta}.$$

Applying relations (41) and (42), we will obtain the Biot constitutive compounds of the medium for each anisotropy of the biphasic medium:

$$\sigma_{ij} = (c_{ijkl} + d_{ijkl})\,\varepsilon_{kl} + \eta_{ijkl}\dot{\varepsilon}_{kl} + (\lambda_{ij} + \beta_{ij})\,\theta \tag{46}$$

and:

$$\sigma = \beta_{ij}\varepsilon_{ij} + \lambda_{ij}\dot{\varepsilon}_{ij} + \gamma\,\theta\,. \tag{47}$$

Equations (46) and (47) may be presented in a simpler way, assuming a new tensor of the skeleton elasticity in the following form:

$$\overset{*}{c}_{ijkl} = c_{ijkl} + d_{ijkl}\,.$$

Even though we do not employ the Onsager symmetry law, we obtain the constitutive compounds that satisfy that law. When only the volumetric deformations of the fluid affect the value of the fluid pressure in pores, which causes that $\beta_{ij} = \beta\delta_{ij}$, and $\lambda_{ij} = \lambda\delta_{ij}$, we obtain a simplified form of the Biot constitutive compounds:

$$\sigma_{ij} = \overset{*}{c}_{ijkl}\varepsilon_{ij} + \eta_{ijkl}\dot{\varepsilon}_{kl} + (\beta + \lambda)\,\delta_{ij}\theta \tag{48}$$

and

$$\sigma = \beta\varepsilon + \lambda\dot{\varepsilon} + \gamma\theta\,. \tag{49}$$

In the case of an isotropic skeleton, the elasticity tensor c_{ijkl} may be expressed by means of two elasticity constants defined by Biot as:

$$\overset{*}{c}_{ijkl} = A\delta_{ij}\delta_{kl} + N(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,. \tag{50}$$

Using the symbols proposed by BIOT in [5], we introduce two new constants: $\beta = Q$ and $\gamma = R$. These constants are supplemented with the skeleton viscosity tensor in the case of an isotropic skeleton, which according to BARTLEWSKA [4] may be expressed by two constants of the shear viscosity and the volumetric viscosity of the skeleton:

$$\eta_{ijkl} = \eta^o \delta_{ij}\delta_{kl} + \eta^p (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{51}$$

where: η^p describes the shear viscosity, and η^o denotes the volumetric viscosity of the skeleton of the porous medium.

Moreover, we will introduce a new constant λ which describes the impact of the velocity of skeleton dilatation on the stress in the fluid. The order of the values in the equations was analysed on the basis of the paper by STRZELECKI et al. [9]. Keeping in line with that work, it is possible to presume that the term with the dilatation velocity of the skeleton is a value of a lesser order than that of the remaining terms. It is possible, therefore, to neglect the influence of the dilatation velocity of the skeleton on the pressure in the fluid. For that reason, in further considerations we will assume that the value of λ is equal to zero.

The constitutive compounds, after the introduction of the above-presented symbols, are equivalent to:

$$\begin{gathered} \sigma_{ij} = 2N\varepsilon_{ij} + 2\eta^p \dot{\varepsilon}_{ij} + (A\varepsilon + (Q+\lambda)\theta)\delta_{ij} + \eta^o \dot{\varepsilon}\delta_{ij}, \\ \sigma = Q\varepsilon + \lambda\dot{\varepsilon} + R\theta. \end{gathered} \tag{52}$$

Assuming that $\lambda = 0$, physical compounds of the Biot model within the Kelvin–Voight rheological framework have the following form (in the case of fan isotropic medium):

$$\begin{gathered} \sigma_{ij} = 2N\varepsilon_{ij} + 2\eta^p \dot{\varepsilon}_{ij} + (A\varepsilon + Q\theta)\delta_{ij} + \eta^o \dot{\varepsilon}\delta_{ij}, \\ \sigma = Q\varepsilon + R\theta. \end{gathered} \tag{53}$$

Constitutive compounds (53) may also be expressed in the form of

$$\begin{cases} \sigma_{ij} = 2N\Psi_k \varepsilon_{ij} + (A\Psi_l \varepsilon + Q\theta)\delta_{ij}, \\ \delta = Q\varepsilon + R\theta, \end{cases}$$

where Ψ are differential operators:

$$\Psi_k = 1 + T_1 \frac{\partial}{\partial t}$$

and

$$\Psi_l = 1 + T_2 \frac{\partial}{\partial t},$$

whereas:

$$T_1 = \frac{\eta^p}{M}, \quad T_1 = \frac{\eta^o}{N}.$$

5. EQUATIONS OF CONSOLIDATION OF BIOT BODY WITHIN KELVIN–VOIGHT RHEOLOGICAL FRAMEWORKS

When the equations of motion include the physical compounds for the Biot body and the rheological framework of the Kelvin–Voight body, then, in the displacements, we obtain the following system of consolidation equations:

$$\begin{cases} N\Psi_k\nabla^2 u_i + (M + N\Psi_l)\varepsilon_{,j} = -\dfrac{H}{R}\sigma_{,j} + \rho_{11}\dfrac{D^s v_i^s}{Dt} + \rho_{12}\dfrac{D^s v_i^l}{Dt} + \rho_{12}\dfrac{D^l v_i^s}{Dt} + \rho_{22}\dfrac{D^l v_i^l}{Dt}, \\ \dfrac{k}{f}\nabla^2\sigma = \dfrac{1}{R}\dot{\sigma} - \dfrac{H}{R}\dot{\varepsilon} + \rho_{12}\dfrac{D^l v_i^s}{Dt} + \rho_{22}\dfrac{D^l v_i^l}{Dt}. \end{cases} \tag{54}$$

When the process is treated as quasi-static, the above system of equations may be written as:

$$\begin{cases} N\Psi_K\nabla^2 u_i + (\Psi_l M + N\Psi_k)\varepsilon_{,i} = -\dfrac{H}{R}\sigma_{,i}, \\ \dfrac{k}{f^2}\nabla^2\sigma = \dfrac{1}{R}\dot{\sigma} - \dfrac{H}{R}\dot{\varepsilon}. \end{cases} \tag{55}$$

When performing the operation of differentiation after the time and after the substitution of $\dfrac{k}{f^2} = K$, the system of equations (55) may be rewritten as follows:

$$\begin{cases} N\Psi_k\nabla^2 u_i + (M + N\Psi_l)\varepsilon_{,i} = -\dfrac{H}{R}\sigma_{,i} + \rho_{11}\dfrac{D^s v_i^s}{Dt} + \rho_{12}\dfrac{D^s v_i^l}{Dt} + \rho_{21}\dfrac{D^l v_i^s}{Dt} + \rho_{22}\dfrac{D^l v_i^l}{Dt}, \\ \dfrac{k}{f}\nabla^2\sigma = \dfrac{1}{R}\dot{\sigma} - \dfrac{H}{R}\dot{\varepsilon} + \rho_{21}\dfrac{D^l v_i^s}{Dt} + \rho_{22}\dfrac{D^l v_i^l}{Dt}. \end{cases} \tag{56}$$

This system of equations describes the process of consolidation as a result of the filtration flow of a viscous Newtonian fluid, seeping through the pores of a viscoelastic skeleton.

6. SUMMARY

Our mathematical model of consolidation may be used for the analysis of the stress and deformation of a soil medium, or the bodies made of porous materials, whose important feature – apart from elasticity – is the viscosity of the skeleton. According to GRYCZMAŃSKI [8] such materials include post-floatation waste and ashes of which dumping grounds or hydrotechnical storages often consist. It may be very important to take into consideration the viscosity of the skeleton of those materials, especially when we want to describe the creep of such buildings or dumping grounds. A vital quality of the model in question is the lack of immediate deformation which is, in our opinion, the major flaw in Biot's model. The process of creep begins always with the zero values of displacement, which is confirmed by laboratory tests.

REFERENCES

[1] AURIAULT J.-L., *Homogenization: Appliciation to Porous Saturated Media in Two Phase Medium Mechanics*, Summer School, Gdańsk, 5–9.09.1983, Publikacje Politechniki Gdańskiej, 1983, 1–56.
[2] AURIAULT J.-L., *Mécanique des Milieux Poreux Saturés Déformables*, Cours de 3 čme cycle MMGE Grenoble, 1986, 1–71.
[3] AURIAULT J.-L., STRZELECKI T., BAUER J., HE S., *Porous deformable media saturated by a very compressible fluid*, Eur. J. Mech. A/Solid, 1990, 9, 4, 373–392.
[4] BARTLEWSKA M., *Określenie parametrów efektywnych modeli reologicznych gruntów spoistych*, PhD Thesis, Wydział Geoinżynierii, Geologii i Górnictwa Politechniki Wrocławskiej, 2009.
[5] BIOT M.A., *General theory of three-dimensional consolidation*, J. Appl. Phys., 1941, No. 12, 155.
[6] BIOT M.A., *General solutions of the equations of elasticity and consolidation of a porous material*, J. Appl. Mech., 1956, 23.
[7] COUSSY O., *Mechanics of Porous Continua*, John Wiley & Sons, 1995.
[8] De GROOT S.R., MAZUR P., *Nonequilibrium Thermodynamics*, North-Holland Publishing Company, 1969.
[9] STRZELECKI T., KOSTECKI S., ŻAK S., *Modelowanie przepływów przez ośrodki porowate*, DWN, Wrocław, 2008.