Studia Geotechnica et Mechanica, Vol. XXXI, No. 2, 2009

PROPAGATION OF THERMAL DISTURBANCE DUE TO THERMAL LOAD IN THERMOELASTICITY WITH THERMAL RELAXATION TIME

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Abstract: In this article, an attempt is made to study the fracas due to a thermal line load in a homogeneous transversely isotropic half-space in linearized theory of generalized thermoelasticity. A combination of the Fourier and the Laplace transform techniques is applied to obtain the solutions of governing equations. The transformed solutions are then inverted using the Cagniard technique for small times. The results obtained theoretically for temperature and stresses are computed numerically for a crystal of zinc, and it is found that variations in stresses and temperature are more prominent at small times and decrease with the passage of time. The results obtained theoretically are represented graphically.

Streszczenie: Podjęto próbę analizy zaburzeń spowodowanych obciążeniem termicznym w jednorodnej, poziomo uwarstwionej półprzestrzeni w ramach uogólnionej liniowej termosprężystości. Zastosowano kombinację transformacji Fouriera i Laplace'a w celu uzyskania rozwiązań równań konstytutywnych. Otrzymane transformaty rozwiązań były następnie odwracane za pomocą procedury Cagniarda dla małych odcinków czasowych. Rozwiązania teoretyczne wykorzystano do numerycznego obliczenia temperatur i naprężeń dla przypadku kryształu cynku. Stwierdzono, że wahania naprężeń i temperatur są większe w małych odcinkach czasowych, a zanikają w miarę upływu czasu.

Резюме: В статье предпринят опыт анализа возмущений, вызванных термической нагрузкой в горизонтально однородном, услоенном полупространстве в рамках обобщенной линейнной термоупругости. Применена комбинация преобвазования Фурье и Лапласа с целью получения решений конститутивных уравнений. Полученные изображения решений были затем обращены при помощи процедуры Каньяра для малых отрезков времени. Теоретические решения были использованы для расчетов температур и напряжений в случае кристаллов цинка. Было установлено, что колебания напряжений и температур больше в малых отрезках времени и устраняются вместе с истеканием времени.

1. INTRODUCTION

The study of thermally induced disturbances in anisotropic bodies is essential for a comprehensive study of their response due to an exposure to temperature fields, which may in turn occur in service or during the manufacturing stages. For example, during the curing stages of filament bound bodies, thermal disturbances may be induced by the heat buildup and cooling processes. The level of these disturbances may well exceed the ultimate strength. The theory of thermoelasticity (NOWACKI [15], [16]) that includes such thermal disturbances has aroused a considerable interest in the last century, but a systematic research started only after thermal waves, called the second sound, that were first measured in such materials as solid helium, bismuth, and sodium fluoride.

Thermoelasticity theories, which admit a finite speed for thermal signals, have been receiving a lot of attention for the past thirty years. The literature dedicated to such theories is quite voluminous and its detailed review can be found in CHANDRASEKHARAIAH [3], [4]. The theories by LORD and SHULMAN [13], GREEN and LINDSAY [8] as well as HETNARSKI and IGNACZAK [12] are among the non-classical theories, which are commonly used for engineering applications.

The Lord and Shulman theory introduces a single time constant to dictate the relaxation of thermal propagation, as well as the rate of change of strain rate, and the rate of change of heat generation. In the Green and Lindsay theory, on the other hand, the thermal and thermo-mechanical relaxations are governed by two different time constants. VERMA and HASEBE [20] studied a problem of dynamic responses with and without energy dissipation in the thermoelastic rotating media.

DHALIWAL and SHERIEF [6] extended [13] the theory of generalized thermoelasticity to anisotropic solids. HAWWA and NAYFEH [11] studied the general problem of thermoelastic waves in anisotropic periodically laminated composites. VERMA and HASEBE [20], [22] studied the wave propagation in plates of general anisotropic media in generalized thermoelasticity. VERMA [18], [19] studied thermoelastic problems by considering equation for anisotropic heat-conducting solids with thermal relaxation time. HARINATH [9], [10] considered the problems of surface point and line source over a homogeneous isotropic thermoelastic half-space in thermoelasticity. De HOOP [5] modified and used a method originally presented by CAGNIARD [2] to solve the disturbances that are generated by an impulsive, concentrated load applied along a line on the free surface of a homogeneous isotropic elastic half-space. NAYFEH and NASSER [14] developed the displacements and temperature fields in a homogeneous isotropic generalized thermoelastic half-space subjected on the free surface to an instantaneously applied heat source using the CAGNIARD–De HOOP [5] method. SHARMA [17] studied the transient generalized thermoelastic waves in a transversely isotropic halfspace, using the same method.

In this paper, the CAGNIARD–De HOOP method is used to study the transient behaviour of homogeneous transversely isotropic linearized thermoelastic material. The motions are caused in the half-space by a thermal line load on its free surface. The thermal relaxation time of the heat conduction is also included in the analysis to ensure that thermal wave speed remains finite. When using the Cagniard–De Hoop method, the strong coupling occurs between thermal and elastic motions, which, however, suggests that we seek solutions for small values of the thermoelastic coupling coefficient. Therefore we express solutions in terms of a small thermoelastic coupling coefficient. Only the approximated short time solutions are considered for thermoelastic response due to the existence of the thermal damping term, which makes the short time solution meaningful. A combination of the Laplace and Fourier transforms is applied to obtain the solutions of governing equations of transversely isotropic thermoelastic solid half-space, which are subjected to thermal line load on its free surface. The resulting equations are then inverted using the Cagniard–De Hoop small times. The results obtained theoretically have been verified numerically and illustrated graphically for single crystal of zinc.

2. BASIC GOVERNING EQUATIONS AND THEIR FORMULATION

The basic field equations of generalized thermoelasticity governing thermoelastic interaction in homogeneous anisotropic solids proposed by DHALIWAL and SHERIEF [6] are:

- Equation of Motion

$$\sum_{j=1}^{3}\left(\frac{\partial \tau_{ij}(\mathbf{u})}{\partial x_j}\right)+f_i=\rho\frac{\partial^2 u_i}{\partial t^2},\quad i=1,2,3. \tag{1}$$

- Energy Equation

$$K_{ij}T_{,ij}-\rho C_e(\dot{T}+\tau_0\ddot{T})=T_0\beta_{ij}(\dot{u}_{i,j}+\tau_0\ddot{u}_{i,j}). \tag{2}$$

- Stress–Strain–Temperature Relations

$$\tau_{ij}=C_{ijkl}e_{kl}-\beta_{ij}T,\quad \beta_{ij}=C_{ijkl}\alpha_{kl}. \tag{3}$$

The summation convention is implied; ρ is the density; t is the time; u_i is the displacement in the x_i direction; K_{ij} are the thermal conductivities; C_e and τ_0 are, respectively, the specific heat at constant strain and thermal relaxation time; σ_{ij} and e_{ij} are the stress and strain tensors, respectively; β_{ij} are thermal moduli; α_{ij} is the thermal expansion tensor; T is the temperature; and the fourth order tensor of the elasticity C_{ijkl} satisfies the (Green) symmetry conditions:

$$c_{ijkl}=c_{klij}=c_{ijlk}=c_{jikl}\quad\text{and}\quad \alpha_{ij}=\alpha_{ji},\quad \beta_{ij}=\beta_{ji},\quad K_{ij}=K_{ji}. \tag{4}$$

Now adjusting equation (1) to (3) for the temperature $T(x, y, z, t)$ and the displacement vector $\mathbf{u}(x, y, z, t) = (u, 0, w)$ in the absence of the body forces and heat source, heat conducting transversely isotropic elastic half-space at the reference temperature T_0 can be presented as follows:

$$\left(C_{11}\frac{\partial^2}{\partial x^2}+C_{44}\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2}\right)u+(C_{13}+C_{44})\frac{\partial^2 w}{\partial x\partial z}=\beta_1\frac{\partial T}{\partial x}, \tag{5}$$

$$\left(C_{44}\frac{\partial^2}{\partial x^2}+C_{33}\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2}\right)w+(C_{13}+C_{44})\frac{\partial^2 u}{\partial x\partial z}=\beta_3\frac{\partial T}{\partial z}, \tag{6}$$

$$K_1\frac{\partial^2 T}{\partial x^2}+K_3\frac{\partial^2 T}{\partial z^2}-\rho C_e\left(\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right)T=T_0\left(\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right)\left(\beta_1\frac{\partial u}{\partial x}+\beta_3\frac{\partial w}{\partial z}\right), \tag{7}$$

where

$$\beta_1=(C_{11}+C_{12})\,\alpha_1+C_{13}\,\alpha_3;\quad \beta_3=2\,C_{13}\,\alpha_1+C_{33}\,\alpha_3, \tag{8}$$

C_{ij} being the elastic parameters, C_e and τ_0 are the specific heat at constant strain and thermal relaxation time, respectively. K_3, K_1 and α_3, α_1 are the coefficients of the thermal conductivities and linear thermal expansions, respectively, along the axis of symmetry and perpendicular to it.

When using the dimensionless quantities (Appendix) in equations, the plane of isotropy is perpendicular to the *z*-axis, which is normal in the half-space. In the beginning the disturbance of undisturbed elastic thermoelastic solid is caused by abruptly applied thermal line load on the free surface. Thermal load applied is symmetrical with respect to the *y*-axis. We consider fixed coordinate system *Oxyz* with origin being any point of the plane boundary $z = 0$. The boundary conditions (on suppressing the primes throughout)

$$\tau_{zz}=0;\quad \tau_{xz}=0;\quad \frac{\partial T}{\partial z}=Q_0\delta(x)f(t),$$

at the surface $z = 0$, become

$$(c_3-c_2)\frac{\partial u}{\partial x}+c_1\frac{\partial w}{\partial z}-\overline{\beta}T=0, \tag{9}$$

$$\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=0, \tag{10}$$

$$\frac{\partial T}{\partial z}=Q_0^*\delta(x)f(t), \tag{11}$$

where $Q_0^*=v_1Q_0/T_0$. The condition at infinity requires that the solutions be bounded as *z* becomes large. Finally, the initial conditions are such that the medium is at rest for $t < 0$.

If we take

$$C_{11}=C_{33}=\lambda+2\,\mu;\quad C_{44}=2\mu;\quad C_{13}=\lambda,$$

$$K_3 = K_1 = K; \quad \alpha_1 = \alpha_3 = \alpha_t; \quad \beta_1 = \beta_3 = (3\lambda + 2\,\mu)\alpha_t, \tag{12}$$

then equations (1) to (3) reduce to the corresponding form for an isotropic body, with Lamé's parameters λ, μ; thermal conductivity K and the coefficients of linear thermal expansion α_t.

3. ANALYSIS

To obtain the solution of the problem following NAYFEH and NASSER [14] and SHARMA [17], we apply the Laplace transform with respect to time and the exponential Fourier transform with respect to the x-coordinate to the system of equations (7) to (11). The appropriate solution of the resulting equation is then constructed and subsequently inverted. The Laplace and the exponential Fourier transforms are defined respectively as:

$$\begin{gathered} L[\phi(x,t)] = \int_0^\infty \phi(x,t)\mathrm{e}^{-pt}dt = \bar{\phi}\,(x,p), \\ F[\bar{\phi}(x,p)] = \int_{-\infty}^\infty \bar{\phi}\,(x,p)\mathrm{e}^{iqx}dx = \hat{\phi}\,(q,p). \end{gathered} \tag{13}$$

With this, a formal solution of equations (7) to (11) is given by

$$(u, w, T) = L^{-1}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{k=1}^{3}(a_{1k}, a_{2k}, a_{3k})\mathrm{e}_{dq}^{-(m_k z - iqx)}\right\}, \tag{14}$$

where L^{-1} designates the inverse Laplace transform and where we have set

$$a_{1j} = \frac{iq\,\bar{f}\,Q_0^*}{c_1 c_2 \varDelta} M_{1j} R_{1j}\,, \tag{15}$$

$$a_{2j} = \frac{\bar{f}\,Q_0^*}{c_1 c_2 \varDelta} M_{1j} m_j S_j\,, \tag{16}$$

$$a_{3j} = -\frac{\bar{f}\,Q_0^*}{\varDelta} M_{1j}(m_j^2 - m_{10}^2)(m_j^2 - m_{20}^2), \quad j = 1, 2, 3\,, \tag{17}$$

$$M_{1i} = \begin{vmatrix} m_k d_k & l_k \\ m_j d_j & l_j \end{vmatrix}, \quad i \neq j \neq k = 1, 2, 3, \tag{18}$$

taking i, j and k in the cyclic order,

$$R_{1j} = (c_1 - c_3\,\overline{\beta}\,)\, m_i^2 - c_2 q^2 - p^2\,,$$

$$S_j = c_2\,\overline{\beta}\; m_j^2 - c_2 q^2 - p^2\,, \quad j = 1, 2, 3,$$

$$\Delta = \sum_{j=1}^{3} m_j\,(m_j^2 - m_{10}^2)(m_j^2 - m_{20}^2) M_{1j}\,, \tag{19}$$

$$l_j = [c_1\, c_2\,\overline{\beta}\;(m_{10}^2 - m_{20}^2) \;+ c_1((c_3 - \;\overline{\beta}\,)q^2 - \;\overline{\beta}\,p^2) - (c_3 - c_2)(c_1 - c_3\,\overline{\beta}\,)q^2]\, m_j^2$$
$$- c_1 c_2\,\overline{\beta}\;\, m_{10}^2 m_{20}^2 \;+ (c_3 - c_2) q^2 (c_2 q^2 + p^2), \tag{20}$$

$$d_j = (c_1 - c_3\,\overline{\beta}\; + c_2\,\overline{\beta}\,)\, m_j^2 - (c_3 - c_2 - \overline{\beta}\,)q^2 - (1 + \overline{\beta}\,)p^2, \quad j = 1, 2, 3, \tag{21}$$

$$m_1^2 + m_2^2 + m_3^2 = m_{10}^2 + m_{20}^2 + m_{30}^2 + \varepsilon_1 \overline{\beta} \tau\, p^2 / \overline{k} c_1\,,$$

$$m_1^2 m_2^2 + m_2^2 m_3^2 + m_3^2 m_1^2 = m_{10}^2 m_{20}^2 + m_{20}^2 m_{30}^2 + m_{30}^2 m_{10}^2 + \frac{\varepsilon_1 \tau\, p^2}{\overline{k} c_1 c_2}\{(c_1 - 2c_3\overline{\beta} + \overline{\beta}^2) q^2 + p^2\}\,,$$

$$m_1^2 m_2^2 m_3^2 = m_{10}^2 m_{20}^2 m_{30}^2 + \varepsilon_1 \frac{\tau (c_2 q^2 + p^2) p^2 q^2}{\overline{k} c_1 c_2}\,, \tag{22}$$

$$m_{10}^2 + m_{20}^2 = (P q^2 + J p^2) / c_1 c_2\,, \tag{23a}$$

$$m_{10}^2 m_{20}^2 = (q^2 + p^2)(c_2 q^2 + p^2) / c_1 c_2\,, \tag{23b}$$

$$m_{30}^2 = (q^2 + \tau\, p^2) / \overline{k}\;, \tag{23c}$$

$$P = c_1 + c_2^2 - c_3^2; \quad J = c_1 + c_2, \quad \tau = \tau_0 + 1/p, \quad \bar{f}(p) = \int_0^\infty f(t) \mathrm{e}^{-pt} dt \tag{23d}$$

Also stresses are given as

$$\tau_{xx} = L^{-1}\left\{ -\frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{k=1}^{3} \;\; (iqa_{1k} + (c_3 - c_2) m_k a_{2k} + a_{3k}) \exp[-(m_k z - iqx)] dq \right\},$$

$$\tau_{zz} = L^{-1}\left\{ -\frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{k=1}^{3} (m_k (c_3 - c_2) a_{2k} + i c_1 q a_{2k} + \overline{\beta} a_{3k}) \exp[-(m_k z - iqx)] dq \right\}, \tag{24}$$

$$\tau_{xz} = L^{-1}\left\{ -\frac{c_2}{2\pi} \int_{-\infty}^{\infty} \sum_{k=1}^{3} (m_k a_{1k} + iqa_{2k}) \exp[-(m_k z - iqx)] dq \right\}.$$

To obtain the solutions that are bounded as $z \to \infty$, we require that m_k have positive real parts. We observe that equation (21) pertains to the coupled dilatational, distortional and thermal waves. To find explicit expression for m_1, m_2 and m_3, we seek solutions of (21) for small values of the thermoelastic coupling ε_1, coupling coefficient between the field of temperature and the field of strain, for the plane state of strain. Since coupling coefficient is a physical characteristic of the material, so the effect of damping and dispersion of thermoelastic waves depends exclusively on the value of this coefficient. The coupling term is generally small for all materials and therefore higher-order terms holding it can be neglected. Neglecting the coupling term simplifies the analysis without noticeable effect on the frequency spectrum as we saw earlier. If we look at the transforms we observe that the expression for m_1, m_2, m_3 and the denominators of a_{jk} are of the higher orders, so the application of inverse transforms is complicated and impractical. On the other hand, by neglecting the terms holding ε_1, inverse transforms can be found from the tables of the Laplace transforms; in that case, however, the sense of the solution will be lost, because the effect of the coupling of the two physical fields will be neglected. We proceed to a solution by linearizing the term holding ε_1, and if the value of the term holding ε_1 is smaller compared with other terms, the expression can be reduced. Therefore following NAYFEH and NASSER [19], assuming that ε_1 is sufficiently small, after the first order approximation in ε_1, we have

$$m_j^2 = m_{j0}^2 + \varepsilon_1 m_{j1}^2 + \ldots, \quad j = 1,\ 2,\ 3, \tag{25}$$

where m_{j0}^2 are given by (22), and

$$m_{j1}^2 = \frac{\tau p^2[(c_2 q^2 + p^2)q^2 - m_{j0}^2\{(c_1 - 2c_3\overline{\beta} + \overline{\beta}^2)q^2 + \overline{\beta}p^2 - c_2\overline{\beta}^2 m_{j0}^2\}]}{\overline{k}c_1 c_2 (m_{j0}^2 - m_{i0}^2)(m_{j0}^2 - m_{k0}^2)}, \quad i \neq j \neq k = 1, 2, 3. \tag{26}$$

If the temperature and strain fields are not coupled with each other, then the thermoelastic coupling constant ε_1 is identically zero. In this case, m_{30} splits from m_{10} and m_{20}. m_{30} from (23c) corresponds to the thermal waves, whereas m_{10} and m_{20} correspond to the coupled longitudinal and transverse elastic waves, and also based on m_{10} and m_{20} it is clear that elastic waves are not affected by thermal variations and thermal relaxation time, but by the anisotropy of the medium.

If the strain and temperature fields are coupled with each other, then from (24) it follows that m_1, m_2 and m_3 get modified due to the thermo-mechanical coupling effects and anisotropy of the medium under study. Equations (15)–(17) with the help of equations (24) yield

$$a_{1j} = \frac{iQ_0^*\bar{f}}{c_1c_2\Delta'}\Bigg[\{(c_1 - c_3\overline{\beta})m_{j0}^2 - c_2q^2 - p^2\}q_{1j}$$

$$+\varepsilon_1\left\{(c_1 - c_3\overline{\beta})m_{j1}^2 g_{1j} + \{(c_1 - c_3\overline{\beta})m_{j0}^2 - c_2q^2 - p^2\}\left(f_{1j} - g_{1j}\frac{\Delta''}{\Delta'}\right)\right\}\Bigg], \qquad (27)$$

$$a_{2j} = \frac{Q_0^*\bar{f}}{c_1c_2\Delta'}\Bigg[(c_2\overline{\beta}m_{j0}^2 - c_2q^2 - p^2)m_{j0}g_{1j}$$

$$+\varepsilon_1\left\{\left[(c_1\overline{\beta}m_{j0}^2 - c_2q^2 - p^2)\left(\frac{m_{j1}^2}{2m_{j0}} - m_{j0}\frac{\Delta''}{\Delta'}\right)\right.\right.$$

$$\left.\left. + c_2\overline{\beta}m_{j1}^2 m_{j0}\right]g_{j1} + (c_2\overline{\beta}m_{j0}^2 - c_2q^2 - p^2)m_{j0}f_{1j}\right\}\Bigg], \quad j = 1, 2, 3, \qquad (28)$$

$$a_{31} = \frac{Q_0^*\bar{f}}{\Delta'}\varepsilon_1(m_{10}^2 - m_{20}^2)m_{11}^2 g_{11}, \qquad (29)$$

$$a_{32} = \frac{-Q_0^*\bar{f}}{\Delta'}\varepsilon_1(m_{20}^2 - m_{10}^2)m_{21}^2 g_{12}, \qquad (30)$$

$$a_{33} = \frac{-Q_0^*\bar{f}}{\Delta'}\Bigg[(m_{30}^2 - m_{10}^2)(m_{30}^2 - m_{20}^2)g_{13} + (m_{30}^2 - m_{10}^2)(m_{30}^2 - m_{20}^2)\left(f_{13} - g_{13}\frac{\Delta''}{\Delta'}\right)$$

$$+((m_{30}^2 - m_{10}^2) + (m_{30}^2 - m_{30}^2))m_{31}^2 g_{13}\Bigg], \qquad (31)$$

where

$$\Delta' = m_{30}(m_{30}^2 - m_{10}^2)(m_{30}^2 - m_{20}^2)g_{13},$$

$$\Delta'' = m_{10}(m_{10}^2 - m_{20}^2)\, m_{11}^2 g_{11} + m_{20}\ (m_{20}^2 - m_{10}^2)\, m_{21}^2\, g_{12}$$
$$+ m_{30}\{(m_{30}^2 - m_{10}^2)(m_{30}^2 - m_{20}^2)f_{13} + (2\, m_{30}^2 - m_{10}^2 - m_{20}^2)\, m_{31}^2\, g_{13}\}, \qquad (32)$$

$$g_{1i} = \begin{vmatrix} m_{k0}D_{k0} & L_{k0} \\ m_{j0}D_{j0} & L_{j0} \end{vmatrix}, \; i \neq j \neq k = 1, 2, 3, \text{ (taking } i, j \text{ and } k \text{ in the cyclic order)},$$

$$f_{1i} = \begin{vmatrix} m_{k0}D_{k0}(\eta) & L_{k1}(\eta) \\ m_{j0}D_{j0}(\eta) & L_{j1}(\eta) \end{vmatrix} + \begin{vmatrix} m_{k0}D_{k1}(\eta) & L_{k0}(\eta) \\ m_{j0}D_{j1}(\eta) & L_{j0}(\eta) \end{vmatrix} + \begin{vmatrix} D_{k0}(\eta)L_{j0}(\eta) & m_{j1}^2/2m_{j0} \\ D_{j0}(\eta)L_{k0}(\eta) & m_{k1}^2/2m_{k0} \end{vmatrix}, \qquad (33)$$

$i \neq j \neq k = 1, 2, 3$ (taking i, j and k in the cyclic order),

$$L_{j0} = Um_{j0}^2 - c_1 c_2 \overline{\beta} m_{10}^2 m_{20}^2 + (c_3 - c_2) q^2 (c_2 q^2 + p^2),$$

$$L_{j1} = Um_{ji}^2,$$

$$D_{j0} = (c_1 - c_3 + c_2) m_{j0}^2 + (c_3 - c_2 - \overline{\beta}) q^2 - (1 + p^2), \tag{34}$$

$$D_{j1} = (c_1 - c_3 \overline{\beta} + c_2 \overline{\beta}) m_{j1}^2, \quad j = 1, 2, 3,$$

$$U = c_1 c_2 \overline{\beta} (m_{10}^2 + m_{20}^2) + c_1 ((c_3 - \overline{\beta}) q^2 - \overline{\beta} p^2) - (c_3 - c_2)(c_1 - c_3 \overline{\beta}) q^2 .$$

4. INVERSION OF TRANSFORMS

Using the Cagniard–De Hoop method to evaluate the right-hand side of equation (24), each integral in (24) is recast into the Laplace transform of a known function, and thus allowing us to write down the inverse transform by inspections. Mathematically this procedure is based on a rather elementary observation that

$$L^{-1}\left\{\frac{p^n}{2\pi}\int_{t_0}^{\infty} f(t)\, \mathrm{e}^{-pt} dt - p^{n-1} f(0) - p^{n-2} f'(0) \ldots - f^{(n-1)}(0)\right\} = \frac{d^n f(t)}{dt^n} H(t - t_0), \tag{35}$$

and

$$L^{-1}\left\{\frac{1}{2\pi p^n}\int_{t_0}^{\infty} f(t)\mathrm{e}^{-pt} dt\right\} = \int_1 \int_2 \int_3 \ldots \int_n f(\overline{t}) H(\overline{t} - t_0) d\overline{t}, \quad n = 0, 1, 2, \ldots \tag{36}$$

A complete description of this technique is given by De HOOP [17], CAGNIARD [18] and FUNG [20]. Making use of this technique, the Laplace transform parameter p is to be isolated as required in (35) and (36). Due to the existence of a damping term in the temperature field equation (9), the isolation of p is impossible (NAYFEH and NASSER [19] and SHARMA [17]). However, this isolation of p may be achieved for small time, i.e. if we assume p to be large. Hence, an expansion in the inverse power of p followed by the change of the variable $q = p\eta$ reduces m_{k0} and m_{k1}^2 to

$$m_{10} = p\alpha_{10}, \quad m_{20} = p\alpha_{20}, \quad m_{30} = p\alpha_{30} + 0.5\,\overline{k}\,\alpha_{30}, \tag{37}$$

$$m_{j1}^2 = p^2[\alpha_{j1}^2 + \alpha_{j1}^{*2} / p], \quad j = 1, 2, 3, \tag{38}$$

$$\alpha_{10}^2, \quad \alpha_{20}^2 = [P\eta^2 + J \pm \{(P\eta^2 + J)^2 - 4c_1 c_2 (\eta^2 + 1)(c_2 \eta^2 + 1)\}^{1/2}] / 2c_1 c_2, \tag{39}$$

$$\alpha_{30}^2 = (\eta^2 + \tau_0)/\bar{k}\ , \tag{40}$$

$$\alpha_{ji}^2 = \tau_0\eta^2(c_2\eta^2+1) - \alpha_{j0}^2\{(c_1 - 2c_3\bar{\beta} + \bar{\beta}^2)\eta^2 + \bar{\beta}^2 - c_2\bar{\beta}^2\alpha_{j0}^2\}/\alpha_{jik}\ , \tag{41}$$

$$\alpha_{jik} = \bar{k}c_1c_2(\alpha_{j0}^2 - \alpha_{k0}^2)(\alpha_{j0}^2 - \alpha_{k0}^2),\quad i \neq j \neq k = 1, 2, 3\ , \tag{42}$$

$$\alpha_{11}^{*2} = \alpha_{11}^2\ \ (\tau_0 + 1/\bar{k}\,(\alpha_{10}^2 - \alpha_{30}^2))\ , \tag{43}$$

$$\alpha_{21}^{*2} = \alpha_{21}^2\ \ (\tau_{0+1}/\bar{k}(\alpha_{20}^2 - \alpha_{30}^2))\ , \tag{44}$$

$$\alpha_{31}^{*2} = \alpha_{31}^2\{\tau_0 + ((\alpha_{10}^2 + \alpha_{20}^2) - 2\alpha_{30}^2)c_1c_2/\alpha_{312}\}$$
$$-\tau_0\{(c_1 - 2c_3\bar{\beta} + \bar{\beta}^2)\eta^2 + \bar{\beta}^2 - 2c_2\ \bar{\beta}^2\ \alpha_{30}^2\}/\bar{k}\alpha_{312}. \tag{45}$$

We take $f(t) = H(t)$ as a unit step function, so that the surface of the half space is subjected to a thermal source of the magnitude of Q_0^* and $\bar{f}(p) = 1/p$. The insertion of equations (35) to (45) into equations (26) to (30) and then into equation (14) yields

$$(u, w, T) = L^{-1}\left(\sum_{i=1}^{3}\bar{u}_i, \sum_{i=1}^{3}\bar{w}_i, \sum_{i=1}^{3}\bar{T}_i\right), \tag{46}$$

$$\bar{u}_k = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{A_{1k}}{p} + \frac{B_{1k}}{p^2}\right)\exp[-p(z\alpha_{k0} + i\eta\, x)d\eta]$$
$$= \frac{1}{\pi}\mathrm{Im}\int_{-\infty}^{\infty}\left(\frac{A_{1k}}{p} + \frac{B_{1k}}{p^2}\right)\exp[-p(z\alpha_{k0} + i\eta\, x)d\eta], \tag{47}$$

$$\bar{w}_k = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{A_{2k}}{p} + \frac{B_{2k}}{p^2}\right)\exp[-p(z\alpha_{k0} + i\eta\, x)d\eta]$$
$$= \frac{1}{\pi}Re\int_{0}^{\infty}\left(\frac{A_{2k}}{p} + \frac{B_{2k}}{p^2}\right)\exp[-p(z\alpha_{k0} + i\eta\, x)d\eta], \tag{48}$$

$$\bar{T}_k = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{A_{3k}}{p} + \frac{B_{3k}}{p^2}\right)\exp[-p(z\alpha_{k0} + i\eta\, x)d\eta]$$
$$= \frac{1}{\pi}Re\int_{0}^{\infty}\left(\frac{A_{3k}}{p} + \frac{B_{3k}}{p^2}\right)\exp[-p(z\alpha_{k0} + i\eta\, x)d\eta], \tag{49}$$

where A_{ij} and B_{ij} can be determined in a straightforward manner.

5. INTEGRAL'S EVALUATION AND SINGULARITY

In the process of evaluating the above integrals from equations (37) to (40), we have to discriminate in favour of (39) for singularities

$$\{(P\eta^2+J)^2-4c_1c_2(\eta^2+1)(c_2\eta^2+1)\}^{1/2}=0 \tag{50}$$

and $\alpha_{k0}=0$, k = 1, 2, 3. It follows that in calculating (46) to (49), taking η as a complex variable and distorting the path of integration in the η-plane, we have arrived at the same poles and branch points:

$$\eta=\pm i,\quad \eta=\pm\frac{i}{\sqrt{c_2}},\quad \eta=\pm i\sqrt{\tau_0},\quad k=1,2,3,$$

as those obtained by NAYFEH and NASSER [19] and SHARMA [17]. For isotropic medium these poles and branch points reduce to

$$\eta=\pm i,\quad \eta=\pm\frac{iv_1}{v_2},\quad \eta=\pm i\sqrt{\tau_0}, \tag{51}$$

where v_1 and v_2 are the respective velocities of dilatational and distortional waves. Again the first equation (50) is a quadratic equation in η^2 and has real roots if the discriminant of this equation is positive. Further, if

$$PJ>2c_1c_2(c_2+1),\qquad P^2>4c_1c_2^2, \tag{52}$$

then equation (50) cannot have positive roots in η^2. Therefore assume that equation (50) is hold and its discriminant is positive, thus the quartic equation has only pure imaginary roots. Other singular points of the integrands are their poles, which are given by

$$(\alpha_{10}^2-\alpha_{20}^2)(\alpha_{20}^2-\alpha_{30}^2)(\alpha_{30}^2-\alpha_{10}^2)=0, \tag{53}$$

$$\alpha_{k0}=0, \tag{54}$$

$$\Delta'(\eta)=0. \tag{55}$$

Equation (53) provides $\alpha_{10}^2=\alpha_{20}^2=\alpha_{30}^2$. This does not hold true as $Re(\alpha_{k0})\geq 0$ and $\alpha_{10}\neq\alpha_{20}\neq\alpha_{30}$, therefore this yields no singularities. The poles of (54) coincide with branch points (51). Now to find the poles given by (55), on taking $\eta = i/V$, rationalizing and simplifying, it reduces to the equation of VERMA [18], giving phase velocity for the isothermal Rayleigh waves in a transversely isotropic half-space in thermoelasticity. It can easily be verified (see ABUBAKAR [21]) that if we assume

$P > J\,c_2$, only one root of the resulting equation (see equation (45), VERMA [18]) satisfies (55) on the upper leaf of the Riemann surface and that is the root which lies in the range of $0 < V^2 < c_2$. Let it is V_R^2, where V_R are the velocities of the Rayleigh waves in uncoupled theory of thermoelasticity, which are same as those obtained by VERMA [9]. Thus for the assumption made, the singularities of integrands (47)–(49), which lie on the upper leaf of the Riemann surface, are

$$\eta = \pm i, \quad \eta = \pm \frac{1}{\sqrt{c_2}}, \quad \eta = \pm i\sqrt{\tau_0}, \quad \eta = \pm\, i\eta_0, \quad \eta = \pm \frac{i}{V_R}. \tag{56}$$

In the special case of $\tau_0 < 1$ and $V_R^2 = 0.1834$, for zinc crystal the path of integration is as long as the real axis. To make the functions of η single valued in the complex plane of integration, we make a cut joining the singularities $i/\sqrt{c_2}$ and $-i/\sqrt{c_2}$ in the η-plane.

First we consider one of the integrals (46)–(49), say

$$\overline{u_1}\,(x,z,p) = \frac{1}{\pi}\ \mathrm{Im} \int_{z/\sqrt{c_2}}^{\infty} \left(\frac{A_{11}}{p} + \frac{B_{11}}{p^2} \right) \frac{d\eta}{dt} .\mathrm{e}^{-pt}\, dt\ . \tag{57}$$

Using equations (35) and (36), we get

$$u_1(x,z,t) = Re\left[\int_0^t A_{11} H\left(\bar{t} - \frac{z}{\sqrt{c_2}} \right) \left(\frac{\partial \eta}{\partial t} \right) d\bar{t} + \int_0^t dt \int_0^{\bar{t}} B_{11} H\left(\bar{t}_1 - \frac{z}{\sqrt{c_2}} \right) \left(\frac{\partial \eta}{\partial t} \right) dt \right]. \tag{58}$$

Similarly

$$u_2(x,z,t) = Re\left[\int_0^t A_{12} H\left(\bar{t} - \frac{z}{\sqrt{c_1}} \right) \left(\frac{\partial \eta_2}{\partial \bar{t}} \right) d\bar{t} + \int_0^t \left\{ \int_0^{\bar{t}} B_{12} H\left(t_1 - \frac{z}{\sqrt{c_1}} \right) \left(\frac{\partial \eta_2}{\partial t_1} \right) dt_1 \right\} dt \right], \tag{59}$$

$$u_3(x,z,t) = Re\left[\int_0^t A_{13} H\left(\bar{t} - z\sqrt{\left(\frac{\tau_0}{\bar{k}}\right)} \right) \left(\frac{\partial \eta_3}{\partial \bar{t}} \right) d\bar{t} + \int_0^t \left\{ \int_0^{\bar{t}} B_{13} H\left(t_1 - z\sqrt{\left(\frac{\tau_0}{\bar{k}}\right)} \right) \left(\frac{\partial \eta_3}{\partial t_1} \right) dt_1 \right\} dt \right]. \tag{60}$$

Thus, we have

$$u_1(x,y,t) = \sum_{k=1}^{3} Re\left[\int_0^t H(\bar{t} + s_k z) \left(\frac{\partial \eta_k}{\partial \bar{t}} \right) d\bar{t} + \int_0^t \left\{ \int_0^{\bar{t}} B_{1k} H(t_1 - s_k z) \left(\frac{\partial \eta_k}{\partial t_1} \right) dt_1 \right\} dt \right], \tag{61}$$

where $s_1 = 1/\sqrt{c_2}$, $s_2 = 1/\sqrt{c_1}$, $s_3 = \sqrt{\tau_0/\bar{k}}$ are the slowness of the transverse dilatational and the thermal waves, respectively.

Similarly

$$w(x,z,t)=\sum_{k=1}^{3}Re\left[\int_0^t A_{2k}H(\bar{t}-s_kz)\left(\frac{\partial\eta_k}{\partial\bar{t}}\right)d\bar{t}+\int_0^t\left\{\int_0^{\bar{t}}B_{2k}H(t_1-s_kz)\left(\frac{\partial\eta_k}{\partial t_1}\right)dt_1\right\}dt\right],\quad(62)$$

$$T(x,z,t)=\sum_{k=1}^{3}Re\left[\int_0^t A_{3k}H(\bar{t}-s_kz)\left(\frac{\partial\eta_k}{\partial t}\right)+\int_0^t B_{3k}H(\bar{t}-s_kz)\left(\frac{\partial\eta_k}{\partial\bar{t}}\right)dt\right],\quad(63)$$

where η_k, k =1, 2, 3, can be determined from $t = \alpha_{k0}z + i\eta_k x$. When the thermoelastic coupling constant ε_1 vanishes, the temperature field also vanishes.

6. NUMERICAL RESULTS AND DISCUSSION

In this section, the results obtained theoretically for temperature and stresses are computed numerically for a single crystal of zinc for which the physical data is given as

$$\varepsilon_1 = 0.0221,\quad c_1 = 0.385,\quad c_2 = 0.2385,\quad c_3 = 0.549,$$

$$c_{11} = 1.628\times10^{11}\ \text{Nm}^{-2},\quad \rho = 7.14\times10^3\ \text{kgm}^{-3},\quad w^* = 5.01\times10^{11}\text{s}^{-1},$$

$$\bar{k} = 1.0,\quad \bar{\beta} = 0.9,\quad \tau_0 = 0.02,\quad T_0 = 296\ \text{K}.$$

The computations were carried out for four values of time namely $\tau = 0.05$, 0.1, 0.2, 0.5 at the surface $z = 0$. The results for temperature (T), horizontal stress (τ_{xx}), vertical stress (τ_{zz}) and shear stress (τ_{xz}) with respect to distance are shown in figures 1, 2, 3 and 4, respectively.

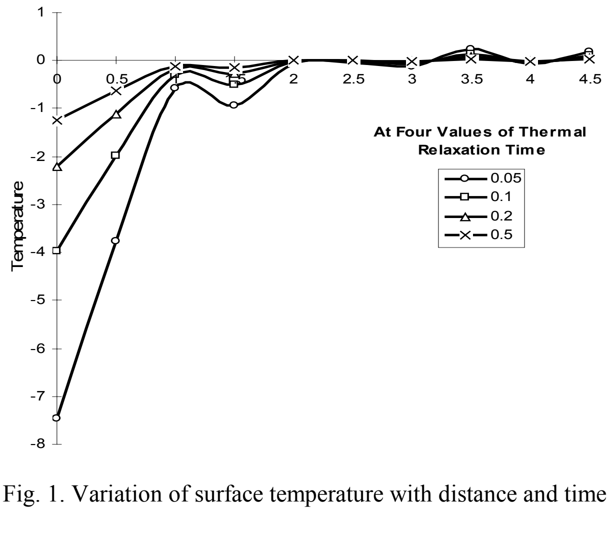

Fig. 1. Variation of surface temperature with distance and time

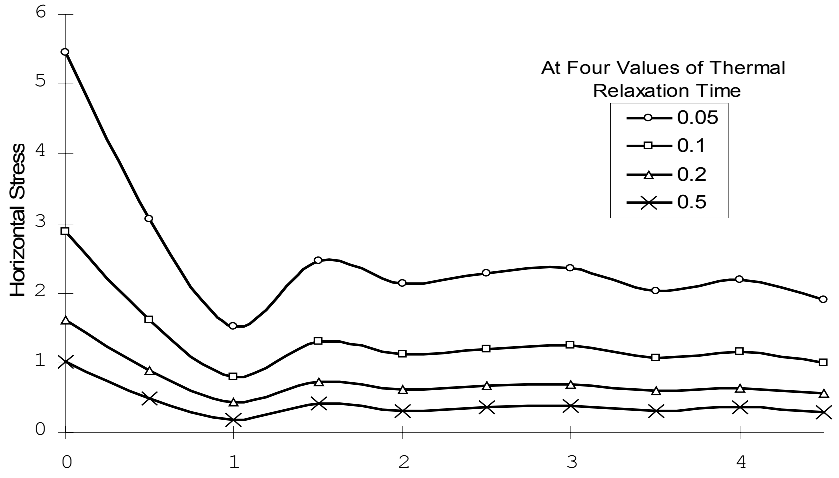

Fig. 2. Variation of horizontal stress with distance and time

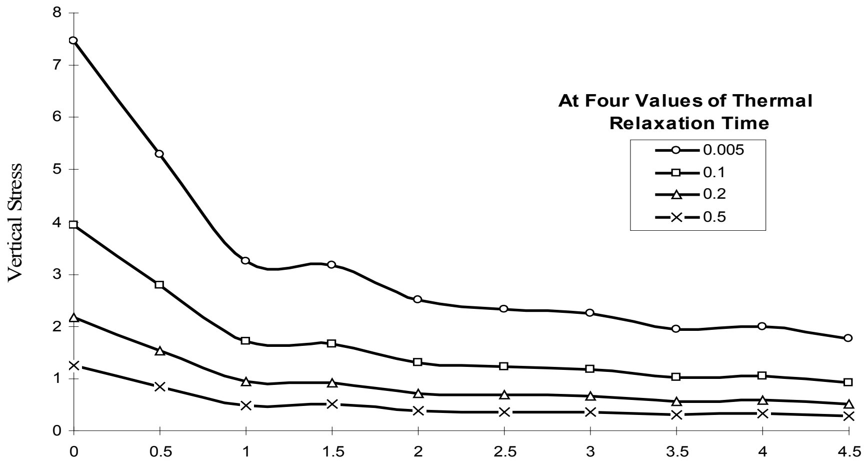

Fig. 3. Variation of vertical stress with distance and time

The figures show that the vertical and shear stresses at the surface are positive and decrease in magnitude with the passage of time, whereas the horizontal stress changes from negative value to positive one with the passage of time. The temperature also increases from negative value to positive value with the passage of time. Also the variations of all these quantities are more prominent at small times and decrease with the passage of time, which established the fact that the second sound effect is short-lived. All these quantities vanish when we move away from the heat source at certain distance at all times, which testifies to the existence of the wave front and ascertains the fact that generalized theory of thermoelasticity admits finite velocity of heat.

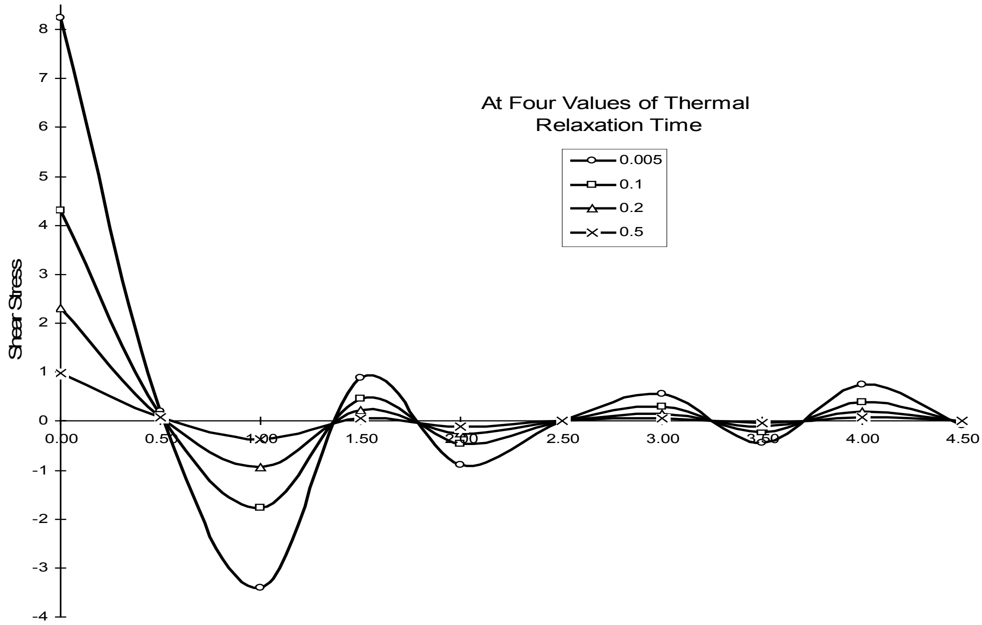

Fig. 4. Variation of shear stress with distance and time

ACKNOWLEDGEMENTS

This work is supported by the Council of Scientific and Industrial Research, Extramural Research Division, CSIR Complex, New Delhi, INDIA, 110012 under grant No. 22(0374)/04/EMR-II.

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APPENDIX

$$x' = \frac{v_1}{k_1}x;\quad z' = \frac{v_1}{k_1}z;\quad t' = \frac{v_1^2}{k_1}t;\quad \tau_0' = \frac{v_1^2}{k_1}\tau_0;\quad u' = \frac{\rho v_1^3}{k_1\beta_1 T_0}u;\quad z' = \frac{\rho v_1^3}{k_1\beta_1 T_0}z;$$

$$T' = \frac{T}{T_0};\quad \bar{k} = \frac{K_3}{K_1};\quad \bar{\beta} = \frac{\beta_3}{\beta_1};\quad c_1 = \frac{C_{33}}{C_{11}};\quad c_2 = \frac{C_{44}}{C_{11}};\tag{6}$$

$$c_3 = \frac{(C_{13}+C_{44})}{C_{11}};\quad \varepsilon_1 = \frac{\beta_1^2 T_0}{\rho C_e \mathrm{v}_1^2},$$

where $k_1 = \dfrac{K_1}{\rho C_e}$ and $v_1 = \left(\dfrac{C_{11}}{\rho}\right)^{1/2}$ are the thermal diffusivity and the velocity of compressional waves in the x-direction, respectively. Here ε_1 is the thermoelastic coupling constant.