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ON THE MAGNETOROTATORY THERMOSOLUTAL CONVECTION (MRTC) OF THE VERONIS TYPE IN THE PRESENCE OF SORET EFFECT

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Abstract: In the present paper, we mathematically prove that the Soret-driven thermosolutal convection of the Veronis type under the simultaneous effect of uniform vertical rotation and magnetic field cannot manifest itself as oscillatory motions of growing amplitude if the thermosolutal Rayleigh number R_s, the Lewis number τ, the Prandtl number σ and the magnetic Prandtl number σ_1 satisfy the inequalities $R_s \le \frac{27\pi^4}{4}\left(1+\frac{\sigma}{\sigma_1}\right)(1-\tau)$ and $\gamma < 1$, γ being the stability ratio.

1. INTRODUCTION

Overstability is a characteristic feature of double diffusive convection and can occur, for example, in a fluid layer with stable solute gradient that is destabilized by raising the temperature of the lower boundary. The linear stability theory for this case is well understood [2], [3] and much information is available about the non-linear development of the instability [4]. Here, overstability depends on the stabilizing effects of the imposed concentration gradient. However, such gradient can also develop in response to the temperature difference applied. This phenomenon, known as the Soret effect [1], arises when the mass flux contains a term that depends on the temperature gradient. The analogous effect that stems from a concentration gradient-dependent term in the heat flux is called the Dufour effect [1] and is important in gases. The phenomenological equations relating the heat flux J_Q and the solute flux J_C to the thermal and solute gradients present in a binary fluid mixture may be formulated (see, for example, de GROOT and MAZUR [5]) as

$$J_Q = K\nabla T - \rho TC\frac{\partial\mu}{\partial C}D'\nabla C\,, \tag{1}$$

$$J_C = -\rho D\{\nabla C + S_T C(1-C)\nabla T\}\,, \tag{2}$$

where T is the temperature, C is the concentration, ρ is the density, K is the thermal conductivity, D is the diffusivity, S_T is the Soret coefficient, $D'(=S_T D)$ is the Dufour coefficient and μ is the chemical potential of the solute. In liquid mixtures, one can neglect the second term in J_Q, the Dufour effect term, but the same approximation cannot be justified in gaseous mixture. On the other hand, the second term in J_C, the Soret effect term, can be significant in both liquid and gaseous mixtures. An externally imposed temperature gradient produces a chemical potential gradient in the system, the normal Soret effect occurs when the concentration of higher molecular mass is higher in the colder region. Similarly, an imposed chemical potential gradient results in a temperature gradient, and the 'normal' Dufour effect is defined by analogy with the Soret effect. The sense of migration of the molecular species is determined by the sign of the Soret coefficient. The rough predictions are as follows:

(a) When the denser component migrates towards the cold plate (positive Soret coefficient), here the upper boundary, we expect the liquid layer to be less stable than in the case of pure liquid.

(b) Migration of the denser component towards the hot plate (negative Soret coefficient), here the lower boundary, we expect the liquid layer to be more stable: the critical Rayleigh number increases.

CALDWELL [6] pointed out the concentration gradient set-up by the Soret diffusion would lead to a situation similar to that considered by VERONIS [2], if the sign of the Soret coefficient S_T were opposite to the normal one. Usually the solute is caused by thermal diffusion to flow from hot to cold. Such diffusion would be destabilizing, and so could not cause an increase in critical Rayleigh number. VERONIS [2], [7] has studied the onset of steady and oscillatory convection generated by infinitesimal perturbations, and has also done calculations on the onset of finite amplitude modes, all with free surface boundary conditions. HURLE and JAKEMAN [8] assumed a salt distribution set-up by thermal diffusion, and included the Soret effect in their perturbation equations as they calculated the onset of steady and oscillatory modes for both free and solid boundaries, for infinitesimal perturbation only. Thus, Hurle and Jakeman included the Soret effect in their equations but Veronis did not. Veronis (and SHIRTELIFFE [9]) used a quantity called R_s, a solute Rayleigh number and for reasonably dilute solution ($c << 1$),

$$R_s = \gamma R_T \quad \text{or} \quad \gamma = \frac{R_s}{R_T}, \tag{3}$$

where R_T is thermal Rayleigh number, the parameter γ is called the stability ratio when applied to thermosolutal or double diffusive phenomenon.

From a geophysical standpoint, the effect of rotation and magnetic field, acting separately or simultaneously, on the present problem is of practical interest. The case where rotation alone is present has been analyzed by ANTORANG and VELARDE [10]. The effect of magnetic field alone on convective instability in a horizontal layer of binary liquid metal has been examined by MASAKI TAKASHIMA [11] and it has been shown that even if a magnetic field is present, the presence of solute plays a prominent role through the Soret effect and that even if the solute is present, the magnetic field inhibits the onset of instability.

In the present investigation, we mathematically establish that the Soret-driven thermosolutal convection of the Veronis type in the presence of uniform vertical rotation and magnetic field cannot manifest itself as oscillatory motions of growing amplitude if the thermosolutal Rayleigh number R_s, the Lewis number τ, the Prandtl number σ and the magnetic Prandtl number σ_1 satisfy the inequalities

$$R_s \le \frac{27\pi^4}{4}\left(1+\frac{\sigma}{\sigma_1}\right)(1-\tau) \quad \text{and} \quad \gamma < 1,$$

γ being the stability ratio.

2. MATHEMATICAL FORMULATION AND ANALYSIS

The relevant governing non-dimensional linearized perturbation equations of Soret-driven thermosolutal convection of the Veronis type in the presence of a uniform vertical rotation and magnetic field with slight change in rotations are given by [10]–[12].

$$(D^2-a^2)\left(D^2-a^2-\frac{p}{\sigma}\right)w = R_T a^2\theta - R_T a^2\phi + TDS - QD(D^2-a^2)h_z\,, \tag{4}$$

$$(D^2-a^2-p)\theta = -w\,, \tag{5}$$

$$\{\tau(D^2-a^2)-p\}\phi + \tau(D^2-a^2)\theta = w\,, \tag{6}$$

$$\left(D^2-a^2-\frac{\sigma_1}{\sigma}\right)h_z = -Dw\,, \tag{7}$$

$$\left(D^2-a^2-\frac{p}{\sigma}\right)\zeta = -Dw - QD\zeta\,, \tag{8}$$

and

$$\left(D^2-a^2-\frac{\sigma_1}{\sigma}\right)\xi = -D\xi\,, \tag{9}$$

where

$$R_T = \frac{g\alpha\beta d^4}{\kappa\upsilon},\ \ \beta>0, \ \text{ and } \ R_s = \frac{g\alpha'\beta' d^4}{\kappa\upsilon},\ \ \beta' = S_T C_0(1-C_0)\beta, \beta'>0,$$

with

$$w = 0 = \theta = \phi = h_z = Dw = \zeta = D\xi \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1, \tag{10}$$

or

$$w = 0 = \theta = \phi = h_z = D^2 w = D\zeta = D\xi \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1, \tag{11}$$

where Z is the real independent variable such that $0 \le Z \le 1$, $D = \frac{d}{dz}$ is the differentiation with respect to Z, $a^2 > 0$ is a constant, $\sigma > 0$ is a constant, $\sigma_1 > 0$ is a constant, $\tau > 0$ is a constant, R_T and R_s are positive constants, $T > 0$ is a constant, $Q > 0$ is a constant, $p = p_r + ip_i$ is a complex constant and as a consequence the dependent variables

$$w(z) = w_r(z) + iw_i(z), \quad \theta(z) = \theta_r(z) + i\theta_i(z), \quad \phi(z) = \phi_r(z) + i\phi_i(z),$$

$$h_z(z) = h_{zr}(z) + ih_{zi}(z), \quad \zeta(z) = \zeta_r(z) + i\zeta_i(z) \quad \text{and} \quad \xi(z) = \xi_r(z) + i\xi_i(z)$$

are complex valued functions of real variable Z. The meaning of symbols from the physical point of view is as follows: Z is the vertical coordinate, $\frac{d}{dz}$ is the differentiation along the vertical direction, a^2 is the square of the wave number, σ is the Prandtl number, σ_1 is the magnetic Prandtl number, τ is the Lewis number, R_T is the thermal Rayleigh number, R_s is the concentration Rayleigh number, T is the Taylor number, Q is the Chandrasekhar number, p is the complex growth rate, w is the vertical velocity, θ is the temperature, ϕ is the concentration, h_z is the vertical magnetic field, ζ is the vertical vorticity, and ξ is the vertical current density. It may further be noted that equations (4)–(11) describe an eigenvalue problem for p and govern the Soret-driven thermosolutal convection of the Veronis type in the presence of uniform vertical rotation and magnetic field for any combination of dynamically free and rigid boundaries.

We prove the following theorem:

Theorem. If $R_T > 0$, $R_s > 0$, $T > 0$, $Q > 0$, $\sigma_1 > \sigma$, $p_r \ge 0$, $p_i \ne 0$ and $R_s \le \frac{27\pi^4}{4}\left(1 + \frac{\sigma}{\sigma_1}\right)(1 - \tau)$, then a necessary condition for the existence of non-trivial solution $(w, \theta, \phi, h_z, \zeta, \xi, p)$ of equations (4)–(9) together with either of the boundary conditions (10) or (11) is that

$$R_s < R_T \quad \text{or} \quad \gamma < 1. \tag{12}$$

Proof. Using the transformations

$$\left.\begin{aligned} \tilde{\phi} &= \left(\frac{1-\tau}{\tau}\right)\phi - \theta \\ \tilde{\theta} &= \theta \\ \tilde{w} &= w \\ \tilde{h}_z &= h_z \\ \tilde{\zeta} &= \zeta \\ \tilde{\xi} &= \xi \end{aligned}\right\}, \tag{13}$$

equations (4)–(11) assume the following forms

$$(D^2 - a^2)\left(D^2 - a^2 - \frac{p}{\sigma}\right)w = R_T a^2\theta - R_s \frac{a^2\tau}{1-\tau}\phi - R_s \frac{a^2\tau}{1-\tau}\theta + TDS - QD(D^2 - a^2)h_z, \tag{14}$$

$$(D^2 - a^2 - p)\theta = -w, \tag{15}$$

$$\left(D^2 - a^2 - \frac{p}{\tau}\right)\phi = \frac{Bw}{\tau}, \tag{16}$$

$$\left(D^2 - a^2 - \frac{p\sigma_1}{\sigma}\right)h_z = -Dw, \tag{17}$$

$$\left(D^2 - a^2 - \frac{p}{\sigma}\right)\zeta = -QD\xi - Dw, \tag{18}$$

and

$$\left(D^2 - a^2 - \frac{p\sigma_1}{\sigma}\right)\xi = -D\zeta \tag{19}$$

with

$$w = \theta = \phi = h_z = Dw = \zeta = D\xi \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1, \tag{20}$$

or

$$w = 0 = \theta = \phi = h_z = D^2 w = D\zeta = D\xi \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1, \tag{21}$$

where $B = \left(\frac{1-2\tau}{\tau}\right) > 0$, and the sign '~' has been omitted for simplicity.

Multiplying equation (14) by w^* (* indicates complex conjugation) throughout and integrating the resulting equation over the vertical range of z, we get

$$\int_0^1 w^*(D^2-a^2)\left(D^2-a^2-\frac{p}{\sigma}\right)wdz = R_T a^2\int_0^1 \theta\, w^* dz - R_s\frac{a^2\tau}{1-\tau}\int_0^1 \theta\, w^* dz$$

$$-R_s\frac{a^2\tau}{1-\tau}\int_0^1 \phi\, w^* dz + T\int_0^1 w^* d\zeta dz - Q\int_0^1 wD(D^2-a^2)h_z dz. \qquad (22)$$

Making use of equations (15)–(19) and the fact that $w(0) = 0 = w(1)$ we can write

$$R_T a^2\int_0^1 \theta\, w^* dz = -R_T a^2\int_0^1 \theta(D^2-a^2-p^*)\theta^* dz\ , \qquad (23)$$

$$R_s\frac{a^2\tau}{1-\tau}\int_0^1 \theta\, w^* dz = -R_s\frac{a^2\tau}{1-\tau}\int_0^1 \theta(D^2-a^2-p^*)\theta^* dz\ , \qquad (24)$$

$$R_s\frac{a^2\tau}{1-\tau}\int_0^1 \phi\, w^* dz = -R_s\frac{a^2\tau^2}{(1-\tau)B}\int_0^1 \phi\left(D^2-a^2-\frac{p^*}{\tau}\right)\phi^* dz\ , \qquad (25)$$

$$T\int_0^1 w^* D\zeta = -T\int_0^1 \zeta D w^* dz = T\int_0^1 \zeta\left(D^2-a^2-\frac{p^*}{\sigma}\right)\zeta^* dz + TQ\int_0^1 \zeta D\xi^* dz$$

$$= T\int_0^1 \zeta\left(D^2-a^2-\frac{p^*}{\sigma}\right)\zeta^* dz + TQ\int_0^1 \xi^*(D\zeta)dz$$

$$= T\int_0^1 \zeta\left(D^2-a^2-\frac{p^*}{\sigma}\right)\zeta^* dz + TQ\int_0^1 \xi^*\left(D^2-a^2-\frac{p^*\sigma_1}{\sigma}\right)\xi dz\ , \qquad (26)$$

and

$$-Q\int_0^1 w^* D(D^2-a^2)h_z dz = Q\int_0^1 (D^2-a^2)h_z D w^* dz$$

$$-Q\int_0^1 (D^2-a^2)h_z\left(D^2-a^2-\frac{p^*\sigma_1}{\sigma}\right)h_z^* dz. \qquad (27)$$

Combining equations (22)–(27) we obtain

$$\int_0^1 w^*(D^2-a^2)\left(D^2-a^2-\frac{p}{\sigma}\right)wdz = -R_T a^2\int_0^1 \theta\left(D^2-a^2-\frac{p^*}{\sigma}\right)\theta^* dz$$
$$+R_s\frac{a^2\tau}{1-\tau}\int_0^1 \theta\left(D^2-a^2-\frac{p^*}{\sigma}\right)\theta^* dz - R_s\frac{a^2\tau^2}{(1-\tau)B}\int_0^1 \phi\left(D^2-a^2-\frac{p^*}{\tau}\right)\phi^* dz$$
$$+T\int_0^1 \zeta\left(D^2-a^2-\frac{p^*}{\sigma}\right)\zeta^* dz + TQ\int_0^1 \xi^*\left(D^2-a^2-\frac{p^*\sigma_1}{\sigma}\right)\xi dz$$
$$-Q\int_0^1 (D^2-a^2)h_z\left(D^2-a^2-\frac{p^*\sigma_1}{\sigma}\right)h_z^* dz\ . \tag{28}$$

Integrating the various terms of equation (28) by parts for an appropriate number of times and making use of either the boundary conditions (20) or (21), it follows that

$$\int_0^1\left(\left|D^2w\right|^2+2a^2|Dw|^2+a^4|w|^2\right)dz+\frac{p}{\sigma}\int_0^1\left(|Dw|^2+a^2|w|^2\right)dz$$
$$=R_T a^2\int_0^1\left(|D\theta|^2+a^2|\theta|^2+p^*|\theta|^2\right)dz - R_s\frac{a^2\tau}{(1-\tau)}\int_0^1\left(|D\theta|^2+a^2|\theta|^2+p^*|\theta|^2\right)dz$$
$$+R_s\frac{a^2\tau^2}{(1-\tau)B}\int_0^1\left(|D\phi|^2+a^2|\phi|^2+\frac{p^*}{\tau}|\phi|^2\right)dz$$
$$-T\int_0^1\left(|D\zeta|^2+a^2|\zeta|^2+\frac{p^*}{\sigma}|\zeta|^2\right)dz - TQ\int_0^1\left(|D\xi|^2+a^2|\xi|^2+\frac{p\sigma_1}{\sigma}|\xi|^2\right)dz$$
$$-Q\int_0^1\left|(D^2-a^2)h_z\right|^2 - Q\frac{p^*\sigma_1}{\sigma}\int_0^1(|Dh_z|^2+a^2|h_z|^2)dz. \tag{29}$$

Equating real and imaginary parts of both sides of equation (29) and cancelling $p_i \neq 0$ throughout from the imaginary part, we get

$$\int_0^1\left(\left|D^2w\right|^2+2a^2|Dw|^2+a^4|w|^2\right)dz+\frac{p_r}{\sigma}\int_0^1\left(|Dw|^2+a^2|w|^2\right)dz$$
$$=R_T a^2\int_0^1\left(|D\theta|^2+a^2|\theta|^2+p_r|\theta|^2\right)dz - R_s\frac{a^2\tau}{(1-\tau)}\int_0^1\left(|D\theta|^2+a^2|\theta|^2+p_r|\theta|^2\right)dz$$

$$+R_s\frac{a^2\tau^2}{(1-\tau)B}\int_0^1\left(|D\phi|^2+a^2|\phi|^2+\frac{p_r}{\tau}|\phi|^2\right)dz-T\int_0^1\left(|D\zeta|^2+a^2|\zeta|^2+\frac{p_r}{\sigma}|\zeta|^2\right)dz$$

$$-TQ\int_0^1\left(|D\xi|^2+a^2|\xi|^2+\frac{p_r\sigma_1}{\sigma}|\xi|^2\right)dz$$

$$-Q\int_0^1\left(\left|\left(D^2-a^2\right)h_z\right|\right)^2dz-Q\frac{\sigma_1 p_r}{\sigma}\int_0^1\left(|Dh_z|^2+a^2|h_z|^2\right)dz \tag{30}$$

and

$$\frac{1}{\sigma}\int_0^1\left(|Dw|^2+a^2|w|^2\right)dz=R_Ta^2\int_0^1|\theta|^2+R_s\frac{a^2\tau}{(1-\tau)}\int_0^1\left(|\theta|^2\right)dz$$

$$-R_s\frac{a^2\tau^2}{(1-\tau)B}\int_0^1\left(|\phi|^2\right)dz-\frac{T}{\sigma}\int_0^1\left(|\zeta|^2\right)dz-TQ\frac{\sigma_1}{\sigma}\int_0^1\left(|\xi|^2\right)dz+Q\frac{\sigma_1}{\sigma}\int_0^1\left(|Dh_z|^2+a^2|h_z|^2\right)dz. \tag{31}$$

We write equation (30) in the alternative form

$$\int_0^1\left(\left|D^2w\right|^2+2a^2|Dw|^2+a^4|w|^2\right)dz+\frac{p_r}{\sigma}\int_0^1\left(|Dw|^2+a^2|w|^2\right)dz$$

$$+R_s\frac{a^2\tau}{(1-\tau)}\int_0^1\left(|D\theta|^2+a^2|\theta|^2\right)dz+T\int_0^1\left(|D\zeta|^2+a^2|\zeta|^2\right)dz$$

$$+TQ\int_0^1\left(|D\xi|^2+a^2|\xi|^2\right)dz+Q\int_0^1\left|\left(D^2-a^2\right)h_z\right|^2dz$$

$$=R_Ta^2\int_0^1\left(|D\theta|^2+a^2|\theta|^2\right)dz+R_s\frac{a^2\tau^2}{(1-\tau)B}\int_0^1\left(|D\phi|^2+a^2|\phi|^2\right)dz$$

$$+p_ra^2\left(R_T\int_0^1|\theta|^2\frac{R_s\tau}{(1-\tau)}-\int_0^1|\theta|^2\frac{R_s\tau}{(1-\tau)B}\int_0^1|\theta|^2-\frac{T}{\sigma_1}\int_0^1|\zeta|^2dz\right.$$

$$\left.-TQ\frac{\sigma_1}{\sigma a^2}\int_0^1|\xi|^2-Q\frac{\sigma_1}{\sigma a^2}\int_0^1|Dh_z|^2+a^2|h_z|^2dz\right) \tag{32}$$

and derive the validity of the theorem from the resulting inequality obtained by replacing each one of the terms of this equation by its appropriate estimates.

We first note that since w,θ,ϕ,h_z and ζ satisfy $w(0)=0=w(1)$, $\theta(0)=0=\theta(1)$, $\phi(0)=0=\phi(1)$, $h_z(0)=0=h_z(1)$ and $\zeta(0)=0=\zeta(1)$, we have by the Rayleigh–Ritz inequality (SCHULTZ [13])

$$\int_0^1 |Dw|^2 dz \geq \pi^2 \int_0^1 |w|^2 dz \,, \tag{33}$$

$$\int_0^1 |D\theta|^2 dz \geq \pi^2 \int_0^1 |\theta|^2 dz \,. \tag{34}$$

$$\int_0^1 |D\phi|^2 dz \geq \pi^2 \int_0^1 |\phi|^2 dz \,, \tag{35}$$

$$\int_0^1 |Dh_z|^2 dz \geq \pi^2 \int_0^1 |h_z|^2 dz \,, \tag{36}$$

and

$$\int_0^1 |D\zeta|^2 dz \geq \pi^2 \int_0^1 |\zeta|^2 dz \,. \tag{37}$$

Further,

$$\int_0^1 |Dw|^2 dz = -\int_0^1 w^* D^2 w dz \leq \left| -\int_0^1 w^* D^2 w dz \right| \leq \int_0^1 \left| w^* D^2 w \right| dz \leq \int_0^1 \left| w^* \right| \left| D^2 w \right| dz$$

$$\leq \int_0^1 |w| \left| D^2 w \right| dz \leq \left(\int_0^1 |w|^2 dz \right)^{1/2} \left(\int_0^1 \left| D^2 w \right|^2 dz \right)^{1/2}$$

(utilizing the Schwartz inequality)

$$\leq \frac{1}{\pi} \left\{ \int_0^1 |Dw|^2 dz \right\}^{1/2} \left\{ \int_0^1 \left| D^2 w \right|^2 dz \right\}^{1/2} . \qquad \text{(using (33))}$$

So that, using equation (33), we have

$$\int_0^1 \left| D^2 w \right|^2 dz \geq \pi^2 \int_0^1 |Dw|^2 dz \geq \pi^4 \int_0^1 |w|^2 dz \,. \tag{38}$$

Therefore by utilizing inequalities (33) and (38), we obtain

$$\int_0^1 \left(\left| D^2 w \right|^2 + a^4 |w|^2 + 2a^2 |Dw|^2 \right) dz \geq (\pi^2 + a^2)^2 \int_0^1 |w|^2 dz \,. \tag{39}$$

Second, since $p_r \geq 0$, we arrive at

$$\frac{p_r}{\sigma} \int_0^1 (|Dw|^2 + a^2 |w|^2) dz \geq 0 \,. \tag{40}$$

Next, multiplying equation (15) by θ^* throughout and integrating the various terms on the left-hand side of the resulting equation by parts for an appropriate number of times by making use of the boundary conditions on θ, namely $\theta(0) = 0 = \theta(1)$, we have from the real part of the final equation

$$\begin{aligned}\int_0^1 (|D\theta|^2 + a^2|\theta|^2)dz + p_r \int_0^1 |\theta|^2 &= \text{real part of } \left(\int_0^1 \theta^* w dz\right)\\ &\le \left|\int_0^1 \theta^* w dz\right|\\ &\le \int_0^1 |\theta^*|\,|w| dz\\ &\le \int_0^1 |\theta|\,|w| dz\\ &\le \left\{\int_0^1 |\theta|^2 dz\right\}^{1/2} \left\{\int_0^1 |w|^2 dz\right\}^{1/2},\end{aligned}$$

utilizing the Schwartz inequality.

Combining this inequality with inequality (34) and the fact that $p_r \ge 0,$ we get

$$(\pi^2 + a^2)\int_0^1 |\theta|^2\, dz \le \left\{\int_0^1 |\theta|^2\, dz\right\}\left\{\int_0^1 |w|^2\, dz\right\},$$

which implies that

$$\left\{\int_0^1 |\theta|^2\, dz\right\}^{1/2} \le \frac{1}{(\pi^2 + a^2)}\left\{\int_0^1 |w|^2\, dz\right\}^{1/2}$$

and thus

$$\int_0^1 (|D\theta|^2 + a^2|\theta|^2)dz \le \frac{1}{(\pi^2 + a^2)}\int_0^1 |w|^2 dz\,. \tag{41}$$

Similarly it follows from equation (16) that

$$\int_0^1 (|D\phi|^2 + a^2 |\phi|^2)dz \le \frac{B}{\tau(\pi^2 + a^2)}\int_0^1 |w|^2\, dz\,. \tag{42}$$

Also, applying the same technique we obtain, since $h_z = 0 = h_z(1)$, the result

$$\int_0^1 | D^2 h_z |^2 \geq \pi^2 \int_0^1 | Dh_z |^2 \, dz \,. \tag{43}$$

With the help of (36) and (43) we obtain

$$\begin{aligned}\int_0^1 \left|(D^2 - a^2)h_z\right|^2 dz &= \int_0^1 \left|D^2 h_z\right|^2 dz + 2a^2 \int_0^1 \left|Dh_z\right|^2 dz + a^4 \int_0^1 \left|h_z\right|^2 dz \\ &\geq \pi^2 \int_0^1 \left|Dh_z\right|^2 + a^2 \int_0^1 \left|Dh_z\right|^2 dz + a^2 \int_0^1 \left|Dh_z\right|^2 dz + \int_0^1 a^4 \left|h_z\right|^2 dz \\ &= (\pi^2 + a^2) \int_0^1 \left|Dh_z\right|^2 + a^2 \left|h_z\right|^2 dz\end{aligned}$$

$$\therefore \; Q \int_0^1 \left|(D^2 - a^2)h_z\right|^2 dz \geq (\pi^2 + a^2) Q \int_0^1 \left|Dh_z\right|^2 + a^2 \left|h_z\right|^2 dz \,. \tag{44}$$

Equation (31) upon using (33) yields the following inequality

$$Q \int_0^1 (Dh_z + a^2 \left|h_z\right|^2) dz > (\pi^2 + a^2) \frac{\sigma}{\sigma_1} \int_0^1 \left|w\right|^2 dz - R_s \frac{a^2 \tau \sigma}{\sigma_1 (1-\tau)} \int_0^1 \left|\theta\right|^2 dz - \frac{T}{\sigma_1} \int_0^1 \left|\zeta\right|^2 dz \,. \tag{45}$$

Therefore, from inequalities (44) and (45), we get

$$\begin{aligned}Q \int_0^1 (D^2 + a^2) \left|h_z\right|^2 dz \geq \frac{(\pi^2 + a^2)^2 \sigma}{\sigma_1} \frac{\sigma}{\sigma_1} \int_0^1 \left|w\right|^2 dz - R_s \frac{a^2 \tau \sigma}{\sigma_1 (1-\tau)} (\pi^2 + a^2) \int_0^1 \left|\theta\right|^2 dz \\ - \frac{T}{\sigma_1} (\pi^2 + a^2) \int_0^1 \left|\zeta\right|^2 dz \,.\end{aligned} \tag{46}$$

Also, from equation (31) and the fact that $p_r \geq 0$, we obtain

$$\begin{aligned}p_r a^2 \Bigg\{ R_T \int_0^1 \left|\theta\right|^2 dz - \frac{R_s \tau}{1-\tau} \int_0^1 \left|\theta\right|^2 dz + \frac{R_s \tau}{(1-\tau)B} \int_0^1 \left|\phi\right|^2 dz - \frac{T}{\sigma a^2} \int_0^1 \left|\varsigma\right|^2 dz \\ - \frac{TQ\sigma_1}{\sigma a^2} \int_0^1 \left|\zeta\right|^2 dz - \frac{Q\sigma_1}{\sigma a^2} \int_0^1 \left|Dh_z\right|^2 + a^2 \left|h_z\right|^2 dz - \Bigg\} \leq 0 \,.\end{aligned} \tag{47}$$

Now, if permissible $R_T \le R_s$ or $\gamma \ge 1$, then we derive from equation (32) and inequalities (34), (39)–(42), (46) and (47) the following inequality

$$\left[(\pi^2+a^2)^2\left\langle 1+\frac{\sigma}{\sigma_1}\right\rangle - \frac{R_s a^2}{(1-\tau)(\pi^2+a^2)}\right]\int_0^1 |w|^2\,dz + \frac{R_s(\pi^2+a^2)\tau}{(1-\tau)}\left(1-\frac{\sigma}{\sigma_1}\right)\int_0^1 |\theta|^2\,dz$$
$$+T(\pi^2+a^2)\left(1-\frac{\sigma}{\sigma_1}\right)\int_0^1 |\zeta|^2\,dz \le 0\,. \tag{48}$$

Therefore inequality (48) implies that

$$R_s > \frac{(\pi^2+a^2)^3}{a^2}\left(1+\frac{\sigma}{\sigma_1}\right)(1-\tau)\,, \tag{49}$$

so that we necessarily have

$$R_s > \frac{27\pi^2}{4}\left(1+\frac{\sigma}{\sigma_1}\right)(1-\tau)\,. \tag{50}$$

Since the minimum value of

$$\frac{(\pi^2+a^2)^3}{a^2} \quad \text{for} \quad a^2>0 \ \text{is} \ \frac{27\pi^4}{4}\,.$$

Hence, if

$$R_s \le \frac{27\pi^4}{4}\left(1+\frac{\sigma}{\sigma_1}\right)(1-\tau)\,,$$

then we must have

$$R_s < R_T \quad \text{or} \quad \gamma < 1\,, \tag{51}$$

and this completes the proof of the theorem.

Theorem 1, from the physical point of view, implies that magnetorotatory thermosolutal convection of the Veronis type in the presence of the Soret effect cannot manifest itself as oscillatory motions of growing amplitude if the thermal Rayleigh number R_s, the Lewis number τ, the Prandtl number σ and the magnetic Prandtl number σ_1 satisfy the inequality

$$R_s \le \frac{27\pi^4}{4}\left(1+\frac{\sigma}{\sigma_1}\right)(1-\tau)$$

and the stability ratio $\gamma < 1$.

Note:

1. It is to be noted here that when both the boundary surfaces are dynamically free the resulting eigenvalue problem described by (14)–(19) together with boundary conditions (20) or (21) can be exactly solved with

$$\zeta = \frac{A\pi}{\pi^2 + a^2 + \dfrac{p}{\sigma}} \cos \pi z \,,$$

where A is an arbitrary constant, and therefore

$$\int_0^1 |D\zeta|^2 \, dz = \pi^2 \int_0^1 |\zeta|^2 \, dz \,,$$

so that inequality (48) again implies (49), (50) and (51) and the theorem is thus proved.

2. In the context of oceanography, $\tau = 0.01$ and $\sigma = 7$ (VERONIS [2]).

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