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A CHARACTERIZATION THEOREM IN HYDROMAGNETIC DOUBLE-DIFFUSIVE CONVECTION COUPLED WITH CROSS DIFFUSIONS

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Abstract: In the present paper, the governing equations of hydromagnetic double-diffusive convection problem of Veronis' type coupled with cross diffusion are linearized by the construction of a proper transformation and the relationship between various energies is established. The analysis made shows that total kinetic energy associated with a disturbance is greater than the sum of its total magnetic and concentration energies in the parameter regime, $\frac{Q\sigma_1}{\pi^2}+\frac{R'_S\sigma}{4\tau^2\pi^4k_2^2}\le 1$, where Q, σ, σ_1, R'_S, and τ, respectively, represent the Chandrasekhar number, the thermal Prandtl number, the magnetic Prandtl number, the modified concentration Rayliegh number and the Lewis number, and k_2 is a constant (to be defined later on). The result is valid for quite general boundary conditions.

1. INTRODUCTION

The stability properties of binary fluids are quite different from those of pure fluids because of the SORET and DUFOUR effects [1], [2]. An externally imposed temperature gradient produces a chemical potential gradient, and the phenomenon known as the Soret effect arises when the mass flux contains a term that depends upon the temperature gradient. The analogous effect that arises from a concentration gradient dependent term in the heat flux is called the Dufour effect. Although it is clear that the thermosolutal and the Soret–Dufour problems are quite closely related, their relationship has never been carefully elucidated. They are, in fact, formally identical and identification is done by means of a linear transformation that takes the equations and boundary conditions for the latter problem into those for the former. The analysis of double diffusive convection becomes complicated in the case where the diffusivity of one property is much greater than the other. Further, when two transport processes take place simultaneously, they interfere with each other and produce cross-diffusion effect. The Soret and Dufour coefficients describe the flux of mass caused by temperature gradient and the flux of heat caused by concentration gradient, respectively. The coupling of the fluxes of the stratifying agents is a prevalent feature in multicomponent fluid systems. In general, the stability of such systems is also affected by the cross-diffusion terms. Generally, it is assumed that the effect of cross diffusions on the stability criteria is negligible. However, there are liquid mixtures for which cross diffusions are of the same order of magnitude as the diffusivities. There are only few studies available on the effect of cross diffusion on double diffusion convection, largely because of the complexity in determining these coefficients. HURLE and JAKEMAN [3] have studied the effect of the Soret coefficient on the double-diffusive convection. They have reported that the magnitude and sign of the Soret coefficient were changed by varying the composition of the mixture. McDOUGALL [4] has made an in depth study of double-diffusive convection, where both Soret and Dufour effects are important.

CHANDRASEKHAR [5] in his investigation of magnetohydrodynamic simple Bénard convection problem sought unsuccessfully the regime in terms of the parameters of the system alone, in which the total kinetic energy associated with a disturbance exceeds the total magnetic energy associated with it, since these considerations are of decisive significance in deciding the validity of the principle of exchange of stabilities. However, the solution for w(= cons tan t(sin πz) is not correct mathematically (and Chandrasekhar was aware of it). BANERJEE and KATYAL until 1985 did not pursue their investigation in this direction and consequently did not see this connection. This gap in the literature on magnetoconvection has been completed by BANERJEE and KATYAL [6] who presented a simple mathematical proof to establish that Chandrasekhar's conjecture is valid in the regime $Q\sigma_1 \leq \pi^2$ and further this result is uniformly applicable for any combination of a dynamically free or rigid boundary when the regions outside the liquid are perfectly conducting or insulating. BANERJEE and KATYAL [6] showed that in the parameter regime

$$\frac{Q\sigma_1}{\pi^2} \leq 1$$

the total kinetic energy associated with a disturbance is greater than the total magnetic energy associated with it.

The present analysis extends this energy consideration to the hydromagnetic double-diffusive convection problem of Veronis' [7] type coupled with cross diffusion. We establish that in the parameter regime

$$\frac{Q\sigma_1}{\pi^2} + \frac{R_s'\sigma}{4\tau^2\pi^4 k_2^2} \leq 1,$$

the total kinetic energy associated with a disturbance is greater than the sum of its total magnetic and concentration energies. A similar characterization theorem for hydromagnetic double-diffusive convection problem coupled with cross diffusion of Stern's [8] type is also established.

2. MATHEMATICAL FORMULATOIN AND ANALYSIS

Here we consider a viscous and finitely heat-conducting fluid statically confined between two horizontal boundaries $z = 0$ and $z = d$ of infinite horizontal extension and finite vertical thickness which are, respectively, maintained at uniform temperatures T_0 and T_1 and concentrations C_0 and C_1 at lower and upper boundaries, respectively, where either $T_0 > T_1$, $C_0 > C_1$ (Veronis' configuration) or $T_0 < T_1$, $C_0 < C_1$ (Stern's configuration) in the presence of a uniform magnetic field acting in a direction opposite to that of gravity. The concentration gradient/temperature gradient thus maintained will, respectively, be qualified as favourable and unfavourable because of their tendencies to decrease/increase the density of the fluid vertically upwards. The extra effect (the Dufour effect and the Soret effect) we consider here is that of the coupled fluxes of the two properties due to irreversible thermodynamic effects.

Let the origin be taken on the lower boundary $z = 0$ with the z-axis perpendicular to it along the vertically upward direction so that the xy-plane then constitutes the horizontal plane $z = 0$. The basic hydrodynamic equations governing the present problem of hydromagnetic double-diffusive convection coupled with cross diffusions are:

$$\frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} + \rho \frac{\partial u_j}{\partial x_j} = 0 . \tag{1}$$

For incompressible fluid

$$\frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} = 0, \tag{2}$$

so that (1) reduces to

$$\frac{\partial u_j}{\partial x_j} = 0 , \tag{3}$$

$$\rho\left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right] = \rho x_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu \frac{\partial u_k}{\partial x_k}\delta_{ij}\right]$$

$$+ \frac{\mu_e H_j}{4\pi}\frac{\partial H_i}{\partial x_j} - \frac{\mu_e}{8\pi}\frac{\partial (\vec{H})^2}{\partial x_i} , \tag{4}$$

$$\frac{\partial}{\partial t}(\rho\, c_v\, T) + \frac{\partial}{\partial x_j}(\rho\, u_j c_v T) = \frac{\partial}{x_j}\left(K \frac{\partial T}{\partial x_j}\right) - p\frac{\partial u_j}{\partial x_j} + \Phi + D_e \frac{\partial^2 C}{\partial x_j^2} , \tag{5}$$

where

$$\Phi = 2\mu e_{ij}^2 - \frac{2}{3}\mu e_{jj}^2 . \tag{6}$$

is the rate at which energy is dissipated by viscosity in each element of the fluid, and e_{ij} is the strain tensor given by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{7}$$

For an incompressible fluid, we have

$$e_{jj} = 0.$$

Thus we have

$$\Phi = 2\mu\, e_{ij}^2 .$$

Making use of equations (1)–(3), we can simplify equation (5) to the following form:

$$\frac{\partial}{\partial t}(\rho\, c_v\, T) + \frac{\partial}{\partial x_j}(\rho u_j c_v T) = \frac{\partial}{\partial x_j}\left(K\frac{\partial T}{\partial x_j}\right) - p\frac{\partial u_j}{\partial x_j} + \Phi + D_e\frac{\partial^2 C}{\partial x_j^2}, \tag{8}$$

$$\frac{\partial C}{\partial t} + u_j\frac{\partial C}{\partial x_j} = \frac{\partial}{\partial u_j}\left(\eta_1\frac{\partial C}{\partial x_j}\right) + S_f\frac{\partial^2 T}{\partial x^2{}_j}, \tag{9}$$

$$\frac{\partial H_i}{\partial t} + \frac{\partial}{\partial x_j}(u_j H_i - u_i H_j) = \eta\nabla^2 H_i . \tag{10}$$

Solonoidal character of magnetic field yields

$$\frac{\partial H_i}{\partial x_i} = 0, \tag{11}$$

$$\rho = \rho_0\{1 + \alpha(T - T_0) - \alpha'(C_0 - C)\} . \tag{12}$$

In the above equations, ρ is the density; t is the time; $x_j\,(j = 1, 2, 3)$ are, respectively, the Cartesian co-ordinates (x, y, z); $u_j\,(= u, v, w)$ are the components of the velocity; $X_i\,(i = 1, 2, 3)$ are the external force components in x, y, z-directions; p is the pressure; μ is the viscosity; C_v is the specific heat at constant volume; T is the temperature; C is the concentration; K is the coefficient of thermal conductivity; η_1 is the coefficient of mass diffusivity; S_f and D_e are coefficients that arise due to the Soret effect and Dufour effect; α and α' are, respectively, the thermal and analogous concentration coefficients of expansion; $H_i\,(= 0, 0, H)$ is the magnetic field; μ_e is the magnetic permeability; η is the magnetic diffusivity.

We now make use of the Boussinesq approximation to simplify the above fundamental system of equations. The essence of this approximation is that the inertial effects produced by density variations are negligible in comparison to the gravitational effects. This implies that ρ can be taken as constant everywhere in the equation of motion except in the term with external force.

Thus, within the framework of the Boussinesq approximation, the fundamental equations governing the present problem take the following form:

$$\frac{\partial u_j}{\partial x_j} = 0, \tag{13}$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial}{\partial x_i}\left(\frac{p}{\rho_0} + \frac{\mu_e |\vec{H}|^2}{8\pi\rho_0}\right) + \nu\nabla^2 u_i + \left(1 + \frac{\partial\rho}{\rho_0} + \frac{\partial\rho'}{\rho_0}\right)X_i + \frac{\mu_e}{4\pi\rho_0} H_j \frac{\partial H_i}{\partial x_j}, \tag{14}$$

$$\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} = \kappa\nabla^2 T + D_f \nabla^2 C, \tag{15}$$

$$\frac{\partial C}{\partial t} + u_j \frac{\partial C}{\partial x_j} = \eta_1\nabla^2 C + S_f \nabla^2 T, \tag{16}$$

$$\frac{\partial H_i}{\partial t} + u_j \frac{\partial H_i}{\partial x_j} = H \frac{\partial u_i}{\partial x_j} + \eta\nabla^2 H_i, \tag{17}$$

$$\frac{\partial H_i}{\partial x_i} = 0, \tag{18}$$

$$\rho = \rho_0\{1 + \alpha(T_0 - T) - \alpha'(C_0 - C)\}, \tag{19}$$

where:

$\delta\rho = \rho_0\alpha(T_0 - T)$,

$\delta\rho' = \rho_0\alpha'(C_0 - C)$,

$\kappa = \dfrac{K}{\rho_0 c_v}$ is the heat diffusivity,

$D_f = \dfrac{D_e}{\rho_0 C_v}$.

The governing equations (13)–(19) yield the following initial stationary state solutions:

$$(u, v, w) = (0, 0, 0),$$

$$T = T_0 - \beta_1 z,$$

$$C = C_0 - \beta_2 z,$$

$$\rho = \rho_0(1+\alpha\beta_1 z - \alpha'\beta_2 z)\,, \tag{20}$$

$$P = P_0 - g\rho_0\left(z + \frac{\alpha\beta_1 z^2}{2} - \frac{\alpha'\beta_2 z^2}{2}\right),$$

$$H = (0,\, 0,\, H),$$

where:

$P = p + \frac{\mu_e |H|^2}{8\pi}$,

$\beta_1 = \frac{T_0 - T}{d}$ is the maintained uniform adverse temperature gradient,

$\beta_2 = \frac{C_0 - C}{d}$ is the uniform favourable concentration gradient.

Let $\delta\rho, (u, v, w), \theta, \phi, \delta p, (h_x, h_y, h_z)$ denote the perturbations in the density ρ, the velocity (0, 0, 0), the temperature *T*, the concentration *C*, the pressure *P* and the magnetic field $\vec{H}$, respectively. Then the linearized perturbation equations are given by:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\,, \tag{21}$$

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}(\delta p) + \mu\nabla^2 u + \frac{\mu_e H}{4\pi}\left(\frac{\partial h_x}{\partial z} - \frac{\partial h_z}{\partial x}\right), \tag{22}$$

$$\rho_0 \frac{\partial v}{\partial t} = -\frac{\partial}{\partial y}(\delta p) + \mu\nabla^2 v + \frac{\mu_e H}{4\pi}\left(\frac{\partial h_y}{\partial z} - \frac{\partial h_z}{\partial y}\right), \tag{23}$$

$$\rho_0 \frac{\partial w}{\partial t} = -\frac{\partial}{\partial z}(\delta p) + \mu\nabla^2 w + g\alpha\rho_0\theta - g\alpha'\rho_0\phi\,, \tag{24}$$

$$\frac{\partial \theta}{\partial t} = \beta_1 w + \kappa\nabla^2\theta + D_f\nabla^2\phi\,, \tag{25}$$

$$\frac{\partial \phi}{\partial t} = \beta_2 w + \eta_1\nabla^2\phi + S_f\nabla^2\theta\,, \tag{26}$$

$$\frac{\partial h_x}{\partial t} = H\frac{\partial u}{\partial z} + \eta\nabla^2 h_x\,, \tag{27}$$

$$\frac{\partial h_y}{\partial t} = H \frac{\partial v}{\partial z} + \eta \nabla^2 h_y , \tag{28}$$

$$\frac{\partial h_z}{\partial t} = H \frac{\partial w}{\partial z} + \eta \nabla^2 h_z , \tag{29}$$

$$\frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} + \frac{\partial h_z}{\partial z} = 0 . \tag{30}$$

Since the fluid under consideration is confined between two horizontal planes $z = 0$ and $z = d$, the variables must satisfy certain boundary conditions on them. Thus, the boundary conditions on w, θ, ϕ and h_z are given by:

$$
\begin{aligned}
&w = 0 = \theta = \phi && \text{on both boundaries,}\\
&\frac{\partial^2 w}{\partial z^2} = 0 && \text{on a tangential stress-free boundary everywhere,}\\
&\frac{\partial w}{\partial z} = 0 && \text{on a rigid boundary,}\\
&h_z = 0 && \text{on both boundaries if the regions outside the fluid are perfectly conducting,}\\
&\left.\begin{aligned} &\frac{\partial h_z}{\partial z} = -a h_z \text{ at } z = d \\ &\frac{\partial h_z}{\partial z} = +\, a h_z \text{ at } z = 0 \end{aligned}\right\} && \text{if the region outside the fluid are insulating.}
\end{aligned}
\tag{31}
$$

We shall now investigate the stability of the system by analyzing an arbitrary perturbation in a complete set of normal modes individually. For the problem in hand, the analysis can be made in terms of two-dimensional periodic waves of assigned numbers. Thus, to all quantities describing the perturbations we ascribe a dependence on x, y and t of the form

$$
\begin{aligned}
&(u, v, w, \theta, \phi, \delta p, h_x, h_y, h_z)\\
&= \{u'(z), v'(z), w'(z), \theta'(z), \phi'(z), \delta p'(z), \delta h_x'(z), \delta h_y'(z), \delta h_z'(z)\}\\
&\cdot \exp(ik_x x + ik_y y + nt),
\end{aligned}
\tag{31$'$}
$$

where k_x, k_y are the wave numbers along the x- and y-directions, respectively, $k = \sqrt{k_x^2 + k_y^2}$ is the resultant wave number and n is the growth rate which is, in general, a complex constant.

Making use of expression (31′), the system of equations (21)–(30) yield the following linearized perturbation equations:

$$ik_x u' + ik_y v' + \frac{dw}{dz} = 0\,, \tag{32}$$

$$\rho_0 n u' = -k_x \delta p' + \mu\left(\frac{d^2}{dz^2} - k^2\right)u' + \frac{\mu_e H}{4\pi}\left(\frac{\partial h'_x}{\partial z} - ik_x h'_z\right), \tag{33}$$

$$\rho_0 n v' = -k_y \delta p' + \mu\left(\frac{d^2}{dz^2} - k^2\right)v' + \frac{\mu_e H}{4\pi}\left(\frac{\partial h'_y}{\partial z} - ik_y h'_z\right), \tag{34}$$

$$\rho_0 n w' = -\frac{d}{dz}\delta p' + \mu\left(\frac{d^2}{dz^2} - k^2\right)w' + g\rho_0 \alpha\theta' - g\rho_0\alpha'\phi'\,, \tag{35}$$

$$n\theta' = \beta_1 w' + k\left(\frac{d^2}{dz^2} - k^2\right)\theta' + D_f\left(\frac{d^2}{dz^2} - k^2\right)\phi'\,, \tag{36}$$

$$n\phi' = \beta_2 w' + \eta_1\left(\frac{d^2}{dz^2} - k^2\right)\phi' + S_f\left(\frac{d^2}{dz^2} - k^2\right)\theta'\,, \tag{37}$$

$$nh'_x = H\frac{du'}{dz} + \eta\left(\frac{d^2}{dz^2} - k^2\right)h'_x\,, \tag{38}$$

$$nh'_y = H\frac{dv'}{dz} + \eta\left(\frac{d^2}{dz^2} - k^2\right)h'_y\,, \tag{39}$$

$$nh'_z = H\frac{\partial w'}{\partial z} + \eta\left(\frac{d^2}{dz^2} - k^2\right)h'_z\,, \tag{40}$$

$$ik_x h'_x + ik_y h'_y + \frac{dh'_z}{dz} = 0\,. \tag{41}$$

Multiplying equation (33) by ik_x and (34) by ik_y, adding the resulting equations and making use of equations (32) and (41), we have

$$\rho_0 n\frac{dw'}{dz} = -k^2\delta p' + \mu\left(\frac{d^2}{dz^2} - k^2\right)\frac{dw'}{dz} + \frac{\mu_e H}{4\pi}\left(\frac{d^2}{dz^2} - k^2\right)h'_z\,. \tag{42}$$

Eliminating $\delta p'$ between equations (42) and (35), we arrive at

$$\left(\frac{d^2}{dz^2} - k^2\right)\left(\frac{d^2}{dz^2} - k^2 - \frac{n}{\nu}\right)w' = \frac{g\alpha k^2\theta'}{\nu} - \frac{g\alpha' k^2\phi'}{\nu} - \frac{\mu_e H}{4\pi\rho_0\nu}\left(\frac{d^2}{dz^2} - k^2\right)\frac{dh'_z}{dz}\,. \tag{43}$$

Further equations (36), (37), (40) can be written as

$$\left(\frac{d^2}{dz^2}-k^2-\frac{n}{\kappa}\right)\theta' = \frac{\beta_1 w'}{\kappa}-\frac{D_f}{\kappa}\left(\frac{d^2}{dz^2}-k^2\right)\phi' , \tag{44}$$

$$\left(\frac{d^2}{dz^2}-k^2-\frac{n}{\eta_1}\right)\phi' = \frac{\beta_2 w'}{\eta_1}-\frac{S_f}{\eta_1}\left(\frac{d^2}{dz^2}-k^2\right)\theta' , \tag{45}$$

$$\left(\frac{d^2}{dz^2}-k^2-\frac{n}{\eta}\right)h_z' = -\frac{H}{\eta}\frac{dw'}{dz} . \tag{46}$$

We shall now introduce the non-dimensional quantities defined by

$$k=\frac{a}{d},\quad \frac{d}{dz}=\frac{D}{d},\quad \hat{w}=\frac{d}{k}w',\quad \hat{\theta}=\frac{\theta'}{\beta_1 d},\quad \hat{\phi}=\frac{\phi'}{\beta_2 d},\quad \hat{p}=\frac{nd^2}{\kappa},$$
$$\sigma=\frac{\nu}{\kappa},\quad \tau=\frac{\eta_1}{\kappa},\quad \sigma_1=\frac{\nu}{\eta},\quad \hat{h}_z=\frac{\eta}{Hk}h_z'. \tag{46$'$}$$

Using the above non-dimensional quantities, omitting caps and dashes for simplicity, the system of equations (42)–(45) assume the following non-dimensional forms:

$$(D^2-a^2)\left(D^2-a^2-\frac{p}{\sigma}\right)w = R_T\, a^2\theta - R_s a^2\phi - QD(D^2-a^2)\,h_z , \tag{47}$$

$$(D^2-a^2-p)\theta + D_T(D^2-a^2)\,\phi = -w , \tag{48}$$

$$\left(D^2-a^2-\frac{p}{\tau}\right)\phi + S_T(D^2-a^2)\,\theta = -\frac{w}{\tau} , \tag{49}$$

and

$$\left(D^2-a^2-\frac{p\sigma_1}{\sigma}\right)h_z = -Dw . \tag{50}$$

The boundary conditions (31) in view of (31′) and (46′) assume the following form:

$w=0=\theta=\phi$	on both boundaries,	
$D^2w=0$	on a tangent stress-free boundary everywhere,	(51)
$Dw=0$	on a rigid boundary,	
$h_z=0$	on both boundaries if the regions outside the fluid are perfectly conducting,	

$$\left.\begin{aligned} Dh_z &= -ah_z \text{ at } z=1 \\ Dh_z &= ah_z \text{ at } z=0 \end{aligned}\right\} \text{ if the regions outside the fluid are insulating.} \quad (51)$$

The meanings of symbols from physical point of view are as follows:

z is the vertical coordinate,

d/dz is the differentiation along the vertical direction,

a^2 is the square of horizontal wave number,

$\sigma = \dfrac{\upsilon}{\kappa}$ is the thermal Prandtl number,

$\sigma_1 = \dfrac{\upsilon}{\eta}$ is the magnetic Prandtl number,

$\tau = \dfrac{\eta_1}{\kappa}$ is the Lewis number,

$R_T = \dfrac{g\alpha\beta_1 d^4}{\kappa\upsilon}$ is the thermal Rayleigh number,

$R_S = \dfrac{g\alpha\beta_2 d^4}{\kappa\upsilon}$ is the concentration Rayleigh number,

$Q = \dfrac{\mu^2 H^2 \sigma d^2}{\rho\upsilon}$ is the Chanderasekhar number,

$D_T = \dfrac{\beta_2 D_f}{\beta_1 \kappa}$ is the Dufour number,

$S_T = \dfrac{\beta_1 S_f}{\beta_2 \eta_1}$ is the Soret number,

ϕ is the concentration,

θ is the temperature,

p is the complex growth rate,

w is the vertical velocity,

h_z is the vertical magnetic field.

In equations (47)–(50), z is real independent variable such that $0 \le z \le 1$, $D = d/dz$ is differentiation w.r.t. z, a^2 is a constant, $\sigma > 0$ is a constant, $\sigma_1 > 0$ is a constant, $\tau > 0$ is a constant, R_T and R_S are the positive constants for the Veronis' configuration and the negative constant for Stern's configuration, $p = p_r + ip_i$ is a complex constant in general such that p_r and p_i are real constants and as a consequence the dependent variables $w(z) = w_r(z) + iw_i(z)$, $\theta(z) = \theta_r(z) + i\theta_i(z)$ and $\phi(z) = \phi_r(z) + i\phi_i(z)$ are complex valued functions (and their real and imaginary parts are real valued).

We now prove the following theorems:

Theorem 1: If $(p, w, \theta, \phi, h_z)$, $p = p_r + ip_i$, $p_r \geq 0$ is a solution of equations (47)–(50) together with boundary conditions (51) with $R_T > 0$, $R_S > 0$ and $\frac{Q\sigma_1}{\pi^2} + \frac{R_s'\sigma}{4\tau^2\pi^4 k_2^2} \leq 1$, then

$$\int_0^1 (|Dw|^2 + a^2|w|^2)dz > Q\sigma_1 \int_0^1 (|Dh_z|^2 + a^2|h_z|^2)dz + R_s' a^2\sigma \int_0^1 |\phi|^2 dz . \quad (52)$$

Proof: We introduce the transformations

$$\begin{aligned} \widetilde{w} &= (S_T + B)\, w , \\ \widetilde{\theta} &= E\theta + F\phi , \\ \widetilde{\phi} &= S_T\theta + B\phi , \\ \widetilde{h}_z &= (S_T + B)\, h_z , \end{aligned} \quad (53)$$

where

$$B = -\frac{1}{\tau}A,$$

$$E = \frac{S_T + B}{D_T + A}A,$$

$$F = \frac{S_T + B}{D_T + A}D_T ,$$

and A is a positive root of the equation

$$A^2 + (\tau - 1)A - \tau\, S_T D_T = 0 .$$

The system of equations (47)–(50) together with boundary conditions (51), upon using the transformation as defined above, takes the following form:

$$(D^2 - a^2)\left(D^2 - a^2 - \frac{p}{\sigma}\right)w = R_T' a^2\theta - R_S' a^2\phi - QD(D^2 - a^2)h_z , \quad (54)$$

$$(k_1(D^2 - a^2) - p)\theta = -w , \quad (55)$$

$$\left(k_2(D^2 - a^2) - \frac{p}{\tau}\right)\phi = -\frac{w}{\tau} , \quad (56)$$

$$\left(D^2 - a^2 - \frac{p\sigma_1}{\sigma}\right)h_z = -Dw, \tag{57}$$

$w = 0 = \theta = \phi$ on both boundaries,

$D^2 w = 0$ on a tangent stress-free boundary everywhere,

$Dw = 0$ on a rigid boundary,

$h_z = 0$ on both boundaries if the regions outside the fluid are perfectly conducting, (58)

$\left.\begin{array}{l} Dh_z = -ah_z \text{ at } z = 1 \\ Dh_z = ah_z \text{ at } z = 0 \end{array}\right\}$ if the regions outside the fluid are insulating,

where

$k_1 = 1 + \frac{\tau\, D_T S_T}{A}$, $k_2 = 1 - \frac{S_T D_T}{A}$ are positive constants,

$R'_T = \frac{(D_T + A)(R_T B + R_S S_T)}{BA - S_T D_T}$, $R'_S = \frac{(S_T + B)(R_S A + R_T D_T)}{BA - S_T D_T}$ are, respectively, the modified thermal Rayleigh number and the modified concentration Rayleigh number. The sign tilde has been omitted for simplicity.

Multiplying equation (57) by h_z^* (the complex conjugate of h_z), integrating the resulting equation over the range of z by parts a suitable number of times, and making use of the boundary conditions (58) we arrive at

$$aM + \int_0^1 (|Dh_z|^2 + a^2 |h_z|^2) dz + \frac{p\sigma_1}{\sigma} \int_0^1 |h_z|^2 dz = -\int_0^1 w\, Dh_z^*, \tag{59}$$

where $M = \{(|h_z|^2)_0 + (|h_z|^2)_1\} \geq 0$.

Equating the real part of equation (59), we obtain

$$aM + \int_0^1 (|Dh_z|^2 + a^2 |h_z|^2) dz + \frac{p_r \sigma_1}{\sigma} \int_0^1 |h_z|^2 dz$$

$$= \text{Real part of} \left(-\int_0^1 w\, Dh_z^*\, dz\right)$$

$$\leq \left|\int_0^1 w\, Dh_z^*\, dz\right|$$

$$\leq \int_0^1 |w||Dh_z| dz$$

$$\leq \left\{\int_0^1 |w|^2\, dz\right\}^{1/2} \left\{\int_0^1 |Dh_z|^2\, dz\right\}^{1/2} \quad \text{(using Schwartz inequality).} \tag{60}$$

Since $p_r \geq 0$, therefore from inequality (60) we have

$$\int_0^1 |Dh_z|^2\, dz < \left\{\int_0^1 |w|^2\, dz\right\}^{1/2} \left\{\int_0^1 |Dh_z|^2 dz\right\}^{1/2} \quad \text{or} \quad \int_0^1 |Dh_z|^2\, dz < \int_0^1 |w|^2\, dz\,. \tag{61}$$

Using inequality (61), it follows from inequality (60) that

$$\int_0^1 (|Dh_z|^2 + a^2 |h_z|^2) dz < \int_0^1 |w|^2\, dz. \tag{62}$$

Since $w(0) = 0 = w(1)$, therefore using the Rayleigh–Ritz inequality [9], we get

$$\int_0^1 |w|^2\, dz < \frac{1}{\pi^2} \int_0^1 |Dw|^2\, dz\,. \tag{63}$$

It follows from inequalities (62) and (63) that

$$\int_0^1 (|Dh_z|^2 + a^2 |h_z|^2) dz < \frac{1}{\pi^2} \int_0^1 |Dw|^2 dz < \frac{1}{\pi^2} \int_0^1 (|Dw|^2 + a^2 |w|^2) dz$$

or

$$Q\sigma_1 \int_0^1 (|Dh_z|^2 + a^2 |h_z|^2) dz + R_s' \sigma a^2 \int_0^1 |\phi|^2\, dz$$
$$< \frac{Q\sigma_1}{\pi^2} \int_0^1 (|Dw|^2 + a^2 |w|^2) dz + R_s' \sigma a^2 \int_0^1 |\phi|^2\, dz\,. \tag{64}$$

Multiplying equation (56) by the complex conjugate of equation (56) and integrating by parts over the vertical range of z for an appropriate number of times and making use of the boundary conditions (58) we have

$$k_2^2 \int_0^1 (|D^2\phi|^2 + 2a^2 |D\phi|^2 + a^4 |\phi|^2) dz + 2 p_r k_2^2 \int_0^1 (|D\phi|^2 + a^2 |\phi|^2)\, dz$$
$$+ \frac{|p|^2}{\tau^2} \int_0^1 |\phi|^2\, dz = \frac{1}{\tau^2} \int_0^1 |w|^2\, dz\,. \tag{65}$$

Since $p_r \geq 0$, therefore from equation (65) it follows

$$\int_0^1 (|D^2\phi|^2 + 2a^2|D\phi|^2 + a^4|\phi|^2)dz < \frac{1}{\tau^2 k_2^2}\int_0^1 |w|^2\,dz\,. \qquad (66)$$

Since $\phi(0) = 0 = \phi(1)$, therefore using the Rayleigh–Ritz inequality [9] we have

$$\pi^2 \int_0^1 |\phi|^2\,dz < \int_0^1 |D\phi|^2\,dz$$

and also

$$\pi^4 \int_0^1 |\phi|^2\,dz \le \int_0^1 |D^2\phi|^2\,dz \text{ (using the Schwartz inequality).} \qquad (67)$$

It follows from inequalities (66) and (67) that

$$(\pi^2 + a^2)^2 \int_0^1 |\phi|^2 dz < \frac{1}{\tau^2 k_2^2}\int_0^1 |w|^2 dz\,,$$

or

$$\frac{(\pi^2 + a^2)^2}{a^2} \int_0^1 |\phi|^2\,dz < \frac{1}{a^2\tau^2 k_2^2}\int_0^1 |w|^2\,dz\,,$$

or

$$a^2 \int_0^1 |\phi|^2 dz < \frac{1}{4\pi^2\tau^2 k_2^2}\int_0^1 |w|^2 dz\,,$$

since the minimum value of $\frac{(\pi^2 + a^2)^2}{a^2}$ for $a^2 > 0$ is $4\pi^2$.

Hence the following inequality results from (63)

$$a^2 \int_0^1 |\phi|^2 dz < \frac{1}{4\pi^4\tau^2 k_2^2}\int_0^1 |Dw|^2 dz < \frac{1}{4\pi^4\tau^2 k_2^2}\int_0^1 (|Dw|^2 + a^2|w|^2)\,dz$$

or

$$R_s' a^2 \sigma \int_0^1 |\phi|^2 dz < \frac{R_s'\sigma}{4\pi^4\tau^2 k_2^2}\int_0^1 (|Dw|^2 + a^2|w|^2)dz\,. \qquad (68)$$

Now from inequalities (64) and (68), we obtain

$$Q\sigma_1\int_0^1(|Dh_z|^2+a^2|h_z|^2)dz+R_s'a^2\sigma\int_0^1|\phi|^2\,dz$$

$$<\left(\frac{Q\sigma_1}{\pi^2}+\frac{R_s'\sigma}{4\tau^2\pi^4k_2^2}\right)\int_0^1(|Dw|^2+a^2|w|^2)dz\ . \tag{69}$$

Therefore, if

$$\frac{Q\sigma_1}{\pi^2}+\frac{R_s'\sigma}{4\tau^2\pi^4k_2^2}\le 1,$$

then from inequality (69) we arrive at

$$\int_0^1(|Dw|^2+a^2|w|^2)dz>Q\sigma_1\int_0^1(|Dh_z|^2+a^2|h_z|^2)\,dz+R_s'a^2\sigma\int_0^1|\phi|^2\,dz\ , \tag{70}$$

and this completes the proof of the theorem.

We note that the left-hand side of equation (70) represents the total kinetic energy associated with a disturbance, while the right-hand side represents the sum of its total magnetic and concentration energies, and Theorem 1 may be stated in the following equivalent form:

At the neutral or unstable state in the hydromagnetic double-diffusive convection problem of the Veronis' type coupled with cross diffusions, the total kinetic energy associated with a disturbance is greater than the sum of its total magnetic and concentration energies in the parameter regime

$$\frac{Q\sigma_1}{\pi^2}+\frac{R_s'\sigma}{4\tau^2\pi^4k_2^2}\le 1$$

and this result is uniformly valid for any combination of dynamically free or rigid boundaries that are either perfectly conducting or insulating.

Theorem 2: If $(p, w, \theta, \phi, h_z)$, $p=p_r+ip_i$, $p_r\ge 0$ is the solution of equations (47)–(50) together with boundary conditions (51) with $R_T<0$, $R_S<0$, and

$$\frac{Q\sigma_1}{\pi^2}+\frac{|R_T'|\sigma}{4\pi^4k_1^2}\le 1\,,$$

then

$$\int_0^1(|Dw|^2+a^2|w|^2)dz>Q\sigma_1\int_0^1(|Dh_z|^2+a^2|h_z|^2)\,dz+|R_T'|a^2\sigma\int_0^1|\theta|^2\,dz\,. \tag{71}$$

Proof. Similar to that of Theorem 1.

We note that the left-hand side of equation (71) represents the total kinetic energy associated with a disturbance, while the right-hand side represents the sum of its total magnetic and thermal energies, and Theorem 2 may be stated in the following equivalent form:

At the neutral or unstable state in the hydromagnetic double-diffusive convection problem of the Stern's type coupled with cross diffusions, the total kinetic energy associated with a disturbance is greater than sum of its total magnetic and thermal energies in the parameter regime

$$\frac{Q\sigma_1}{\pi^2}+\frac{|R_T'|\sigma}{4\pi^4k_1^2}\leq 1$$

and this result is uniformly valid for any combination of dynamically free or rigid boundaries that are either perfectly conducting or insulating.

3. CONCLUSIONS

In the present paper, the hydromagnetic double-diffusive convection problem of the Veronis' and Stern's type configuration coupled with cross diffusion is considered. The investigation of cross diffusion effect is motivated by its interesting complexities as a double-diffusive phenomenon which has its importance in various fields such as high-quality crystal production, oceanography, production of pure medication, solidification of molten alloys, exothermally heated lakes and magmas. The analysis made brings out the following main conclusions:

(i) At the neutral or unstable state in the magnetohydrodynamic double-diffusive convection problem of the Veronis' type coupled with cross diffusion, the total kinetic energy associated with a disturbance is greater than the sum of its total magnetic and concentration energies in the parameter regime

$$\frac{Q\sigma_1}{\pi^2}+\frac{R_s'\sigma}{4\tau^2\pi^4k_2^2}\leq 1$$

and this result is uniformly valid for any combination of dynamically free or rigid boundaries that are either perfectly conducting or insulating.

(ii) At the neutral or unstable state in the hydromagnetic double-diffusive convection problem of the Stern's type coupled with cross diffusions, the total kinetic energy associated with a disturbance is greater than sum of its total magnetic and thermal energies in the parameter regime

$$\frac{Q\sigma_1}{\pi^2}+\frac{|R_T'|\sigma}{4\pi^4k_1^2}\leq 1$$

and this result is uniformly valid for any combination of dynamically free or rigid boundaries that are either perfectly conducting or insulating.

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