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STABILITY OF STRATIFIED ELASTICO-VISCOUS RIVLIN–ERICKSEN ROTATING FLUID IN THE PRESENCE OF UNIFORM HORIZONTAL MAGNETIC FIELD IN POROUS MEDIUM

S.K. KANGO, VIKRAM SINGH

Department of Mathematics, Government College, Haripur (Manali)
Himachal Pradesh – 175136 India. drsanjaykanungo@in.com

Abstract: The stability of stratified elastico-viscous Rivlin–Ericksen fluid in the presence of horizontal magnetic field and rotation in porous medium is considered. In contrast to the Newtonian fluids, the system is found to be unstable at stable stratification for low values of permeability or high values of kinematic viscoelasticity. Magnetic field is found to stabilize the small wavelength perturbations for unstable stratification. It has been found that the growth rate increases with the increase in kinematic viscosity and permeability, whereas it decreases with the increase in kinematic viscoelasticity.

LIST OF SYMBOLS

P	– pressure,
ρ	– fluid density,
δp	– the perturbation in pressure,
$\delta\rho$	– the perturbation in density,
μ	– coefficient of viscosity,
μ'	– coefficient of viscoelasticity,
ν	– kinematic viscosity,
ν'	– kinematic viscoelasticity,
ε	– medium porosity,
k_1	– medium permeability,
$\vec{g}\,(0, 0, -g)$	– acceleration due to gravity,
$\vec{H}\,(H, 0, 0)$	– magnetic field vector having components $(H, 0, 0)$,
η	– electrical resistivity,
μ_e	– magnetic permeability,
$\vec{\Omega}\,(\Omega, 0, 0)$	– rotation vector having the components $(\Omega, 0, 0)$,
κ	– thermal diffusivity,
$\vec{q}\,(u, v, w)$	– perturbation in fluid velocity $\vec{g}\,(0, 0, 0)$,
$\vec{h}\,(h_x, h_y, h_z)$	– perturbation in magnetic field $\vec{H}\,(H, 0, 0)$,
β	– a constant,
∇	– del operator,
δ	– perturbation in the respective physical quantity,

π	– constant value,
∂	– curly operator,
i	– square root of (–1),
k_x, k_y	– wave numbers in the x- and y-directions, respectively,
$k = (k_x^2 + k_y^2)^{1/2}$	– wave number of the disturbance,
n	– growth rate of the disturbance,
V_A^2	– square of the Alfvén velocity,
d	– depth of the fluid layer,
s	– an integer.

1. INTRODUCTION

A detailed account of the theoretical and experimental results of the thermal instability (Benard convection) in an incompressible, viscous (Newtonian) fluid layer, under varying assumptions of hydrodynamics and hydromagnetics, has been given in the celebrated monograph by CHANDRASEKHAR [1]. OLDROYD [2] has studied some steady flows of a general elastico-viscous liquid while the effect of surface tension and viscosity on the stability of two superposed fluids has been studied by REID [3]. SHARMA and GUPTA [12] have also studied the stability of stratified elastico-viscous Walters' B′ fluid in the presence of horizontal magnetic field and rotation.

The medium has been considered to be non-porous in all the above studies. WOODING [13] has considered the Rayleigh instability of thermal boundary layer in flow through porous medium whereas SHARMA [6] has studied thermal instability of a viscoelastic fluid in hydromagnetics. SHARMA et al. [11] have also studied the stability of stratified elastico-viscous Walters' B′ fluid in the presence of horizontal magnetic field and rotation in porous medium. There are many elastico-viscous fluids that cannot be characterized by Maxwell's constitutive relations or Oldroyd's constitutive relations. Rivlin–Ericksen fluid is one such class of elastico-viscous fluids. RIVLIN and ERICKSEN [4] have proposed a theoretical model for such another elastico-viscous class.

The flow through porous medium has been of considerable interest in recent years, particularly in geophysical fluid dynamics. A porous medium is a solid with holes in it, and is characterized by the manner in which the holes are imbedded, how they are interconnected and the description of their location, shape and interconnection. However, the flow of a fluid through a homogeneous and isotropic porous medium is governed by Darcy's law which states that the usual viscous term in the equations of stratified Rivlin–Ericksen fluid motion is replaced by the resistance term $\left[-\frac{1}{k_1}\left(\mu + \mu' \frac{\partial}{\partial t}\right)\right] q$, where μ and μ' are the viscosity and viscoelasticity of the incompressible Rivlin–Ericksen fluid, k_1 is the medium permeability and q is the Darcian (filter) velocity of the fluid. SHARMA [7] has studied the effect of uniform magnetic field and uniform rotation on the stability of two superposed fluids in porous medium. SHARMA and KUMAR [5] have studied the steady flow and heat transfer of Walters' B′ fluid through porous pipe of uniform circular cross-section with small suction. SHARMA and KUMAR [9] have also studied Rayleigh Taylor instability of two superposed conducting Walters' B′ fluid in hydromagnetics whereas SHARMA and KANGO [10] have studied the thermal convection in Rivlin–Ericksen elastico-viscous fluid in porous medium in hydromagnetics.

Keeping in mind the growing importance of non-Newtonian fluids in modern technology, industry, chemical technology and dynamics of geophysical fluids and considering the conflicted tendencies of magnetic field and rotation while acting together, our interest, in the present paper is to study the stability of stratified elastico-viscous Rivlin–Ericksen rotating fluid in the presence of uniform horizontal magnetic field in porous medium.

2. FORMULATION OF THE PROBLEM AND PERTURBATION EQUATIONS

The initial stationary state whose stability we wish to examine is that of an incompressible, infinitely conducting Rivlin–Ericksen fluid of variable density, kinematic viscosity and kinematic viscoelasticity, arranged in horizontal strata in a porous medium of variable porosity and medium permeability. We are considering little unusual configuration in which not only the magnetic field is parallel to the layer but rotation is also oriented in the same direction, i.e., the elastico-viscous fluid is acted on by gravity force $\vec{g} = (0, 0, -g)$, a uniform horizontal magnetic field $\vec{H} = (H, 0, 0)$ and a uniform horizontal rotation $\vec{\Omega} = (\Omega, 0, 0)$. The same type of configuration has been considered by SHARMA [6] while studying the stability of a stratified fluid in porous medium in the presence of horizontal magnetic field and rotation.

Consider an infinite horizontal layer of thickness d bounded by the planes $z = 0$ and $z = d$. The character of the equilibrium of this stationary state is determined by supposing that the system is slightly disturbed and then, following its further evolution.

Let p, ρ, ε, μ_e, μ, μ' and $\vec{q}\,(u, v, w)$ denote, respectively, fluid pressure, density, medium porosity, magnetic permeability, viscosity, viscoelasticity and the filter velocity of fluid (initially zero). Then the hydromagnetic equations relevant to the problem are

$$\frac{\rho}{\varepsilon}\left[\frac{\partial \vec{q}}{\partial t} + \frac{1}{\varepsilon}(q \cdot \nabla)\vec{q}\right] = -\nabla p + \vec{q}\rho + \frac{\mu_e}{4\pi}(\nabla \times \vec{H}) \times \vec{H} - \frac{1}{k_1}\left(\mu + \mu'\frac{\partial}{\partial t}\right)\vec{q} + \frac{2\rho}{\varepsilon}(\vec{q} \times \vec{\Omega}), \quad (1)$$

$$\nabla \cdot \vec{q} = 0\,, \quad (2)$$

$$\varepsilon \frac{\partial \rho}{\partial t} + (\vec{q} \cdot \nabla)\rho = 0 , \tag{3}$$

$$\nabla \cdot \vec{H} = 0 , \tag{4}$$

$$\varepsilon \frac{\partial \vec{H}}{\partial t} = (\vec{H} \cdot \nabla)\vec{q} . \tag{5}$$

Equation (3) represents the fact that the density of a particle remains unchanged as we follow it with its motion.

Let δp, $\delta \rho$, $\vec{q} = (u, v, w)$ and $\vec{h} = (h_x, h_y, h_z)$ denote the perturbations in pressure p, density ρ, velocity (0, 0, 0) and horizontal magnetic field $\vec{H} = (H, 0, 0)$, respectively. Then the linearized hydromagnetic perturbation equations become

$$\frac{\rho}{\varepsilon}\frac{\partial \vec{q}}{\partial t} = -\nabla \delta p + \vec{g}\delta\rho + \frac{\mu_e}{4\pi}(\nabla \times \vec{h}) \times \vec{H} - \frac{1}{k_1}\left(\mu + \mu' \frac{\partial}{\partial t}\right)\vec{q} + \frac{2\rho}{\varepsilon}(\vec{q} \times \vec{\Omega}), \tag{6}$$

$$\nabla \cdot \vec{q} = 0 , \tag{7}$$

$$\varepsilon \frac{\partial \delta\rho}{\partial t} + (\vec{q} \cdot \nabla)\rho = 0 , \tag{8}$$

$$\nabla \cdot \vec{h} = 0 , \tag{9}$$

$$\varepsilon \frac{\partial \vec{h}}{\partial t} = (\vec{H} \cdot \nabla)\vec{q} . \tag{10}$$

Analyzing the disturbances into normal modes, we seek solutions whose dependence on x, y, z and time t is given by

$$f(z)\exp(ik_x + ik_y + nt) , \tag{11}$$

where $f(z)$ is some function of z and k_x, k_y are the wave numbers in the x- and y-directions, $k = \sqrt{k_x^2 + k_y^2}$ is the resultant wave number and n is the growth rate of the disturbance which is , in general, a complex constant.

Equations (6)–(10), using expression (11) in the Cartesian coordinates become

$$\frac{\rho}{\varepsilon} nu = -ik_x \delta p - \frac{1}{k_1}(\mu + \mu' n)u , \tag{12}$$

$$\frac{\rho}{\varepsilon} n v = -ik_y \delta p - \frac{1}{k_1}(\mu + \mu' n)v + \frac{\mu_e H}{4\pi}(ik_x h_y - ik_y h_x) + \frac{2}{\varepsilon}\rho \Omega w , \tag{13}$$

$$\frac{\rho}{\varepsilon}nw = -D\delta p - \frac{1}{k_1}(\mu + \mu' n)w + \frac{\mu_e H}{4\pi}(ik_x h_z - Dh_x) - \frac{2}{\varepsilon}\rho\Omega v - g\delta\rho, \tag{14}$$

$$ik_x u + ik_y v + Dw = 0, \tag{15}$$

$$\varepsilon n\delta\rho + w(D\rho) = 0, \tag{16}$$

$$ik_x h_x + ik_y h_y + Dh_z = 0, \tag{17}$$

$$\varepsilon n h_x = ik_x H u, \tag{18}$$

$$\varepsilon n h_y = ik_x H v, \tag{19}$$

$$\varepsilon n h_z = ik_x H w, \tag{20}$$

where D stands for d/dz.

Eliminating u, v and δp from equations (12)–(14) and using equations (15)–(20), we get

$$\rho\left[n^2 + \frac{\varepsilon n}{k_1}(v + v'n) + k_x^2 V_A^2\right]D^2 w + n^2(D\rho)(Dw)$$

$$-\left[k^2\left(n^2 + \frac{\varepsilon n}{k_1}(v + v'n) + k_x^2 V_A^2\right)\rho + \frac{4\rho n^2 \Omega^2 k_x^2}{n^2 + \frac{\varepsilon n}{k_1}(v + v'n) + k_x^2 V_A^2}\right]w$$

$$+ 2ink_y\Omega(D\rho)w = 0, \tag{21}$$

where $v = \mu/\rho$, $v' = \mu'/\rho$ and $V_A^2 = \mu_e H^2/4\pi\rho$ (square of Alfvén velocity).

Equation (21) is the general equation to consider the stability of stratified Rivlin–Ericksen fluid in a porous medium in the presence of horizontal magnetic field and uniform rotation. In the absence of viscoelasticity, i.e., ($v' = 0$), equation (21) reduces to the result by SHARMA [7].

3. THE CASE OF EXPONENTIALLY VARYING STRATIFICATIONS

Let us assume the stratifications in density, viscosity, viscoelasticity, medium porosity and medium permeability of the forms

$$\rho = \rho_0 e^{\beta z}, \quad \mu = \mu_0 e^{\beta z}, \quad \mu' = \mu_0' e^{\beta z}, \quad \varepsilon = \varepsilon_0 e^{\beta z}, \quad k_1 = k_{10} e^{\beta z}, \tag{22}$$

where ρ_0, μ_0, μ_0', ε_0, k_{10} and β are constants. Equation (22) implies that the kinematic viscosity $\nu_0\left(=\frac{\mu}{\rho}=\frac{\mu_0}{\rho_0}\right)$ and the kinematic viscoelasticity $\nu_0'\left(=\frac{\mu'}{\rho}=\frac{\mu_0'}{\rho_0}\right)$ are constant everywhere. Using stratification of the form (22), equation (21) transforms to

$$\left[n^2+\frac{\varepsilon_0 n}{k_{10}}(\nu_0+\nu_0' n)+k_x^2V_A^2\right]D^2w+n^2\beta(Dw)$$

$$-\left[k^2\left(n^2+\frac{\varepsilon_0 n}{k_{01}}(\nu_0+\nu_0' n)+k_x^2V_A^2\right)+\frac{4n^2\Omega^2k_x^2}{n^2+\frac{\varepsilon_0 n}{k_{10}}(\nu_0+\nu_0' n)+k_x^2V_A^2}-g\beta k^2\right]w=0. \qquad (23)$$

The general solution of equation (23) is given by

$$w=A_1e^{m_1z}+A_2e^{m_2z}, \qquad (24)$$

where A_1, A_2 are two arbitrary constants and m_1, m_2 are the roots of the equation

$$\left[n^2+\frac{\varepsilon_0 n}{k_{10}}(\nu_0+\nu_0' n)+k_x^2V_A^2\right]m^2+n^2\beta m$$

$$-\left[k^2\left(n^2+\frac{\varepsilon_0 n}{k_{01}}(\nu_0+\nu_0' n)+k_x^2V_A^2\right)+\frac{4n^2\Omega^2k_x^2}{n^2+\frac{\varepsilon_0 n}{k_{10}}(\nu_0+\nu_0' n)+k_x^2V_A^2}-g\beta k^2\right]=0. \qquad (25)$$

Here, we consider the fluid to be confined between two rigid planes $z = 0$ and $z = d$.

The boundary conditions for the case of two rigid surfaces are

$$w=0 \quad \text{at} \quad z=0 \quad \text{and} \quad z=d. \qquad (26)$$

The vanishing of w at $z = 0$ is satisfied by the choice

$$w=A(e^{m_1z}-e^{m_2z}), \qquad (27)$$

while vanishing of w at $z = d$ requires $e^{(m_1-m_2)d} = 1$, i.e.,

$$(m_1-m_2)d=2is\pi, \qquad (28)$$

where s is an integer. Solving (25), we get

$$m_{1,2}=\frac{1}{2}\left[\frac{-n^2\beta}{L_2}\pm\left\{\frac{n^4\beta^2}{L_2^2}+4\left(k^2+\frac{4n^2\Omega^2k_x^2}{L_2^2}-\frac{g\beta k^2}{L_2^2}\right)\right\}^{1/2}\right], \qquad (29)$$

where $L_2 = \left(n^2 + \frac{\varepsilon_0 n}{k_{10}}(\nu_0 + \nu_0' n) + k_x^2 V_A^2 \right)$.

Inserting the values of m_1, m_2 from equation (29) in equation (28), we get

$$n^4\left[\frac{\beta^2 d^2}{k^2 d^2 + s^2\pi^2} + 4\left(1 + \frac{2\varepsilon_0 \nu_0'}{k_{10}} + 4\frac{\varepsilon_0^2 \nu_0'^2}{k_{10}}\right)\right] + n^3\left[\frac{\varepsilon_0 \nu_0}{k_{10}}\left(1 + \frac{\varepsilon_0 \nu_0'}{k_{10}}\right)\right]$$
$$+ n^2\left[4\frac{\varepsilon_0^2 \nu_0^2}{k_{10}^2} + 2k_x^2 V_A^2\left(1 + \frac{4\varepsilon_0 \nu_0'}{k_{10}}\right) + \frac{4\Omega^2 V_A^2 d^2}{k^2 d^2 + s^2\pi^2} - \frac{g\beta k^2 d^2}{k^2 d^2 + s^2\pi^2}\left(1 + \frac{4\varepsilon_0 \nu_0'}{k_{10}}\right)\right]$$
$$+ n\left[\frac{4\varepsilon_0 \nu_0}{k_{10}}\left\{2k_x^2 V_A^2 - \frac{g\beta k^2 d^2}{k^2 d^2 + s^2\pi^2}\right\}\right] + 4k_x^2 V_A^2\left\{k_x^2 V_A^2 - \frac{g\beta k^2 d^2}{k^2 d^2 + s^2\pi^2}\right\} = 0. \tag{30}$$

Equation (30) is biquadratic in n, therefore, it must give four roots and it is the dispersion relation for studying the effects of rotation and horizontal magnetic field on the stability of stratified elastico-viscous (exponentially varying density) fluid in porous medium.

4. DISCUSSION AND CONCLUSIONS

If $\beta < 0$ (stable stratification), equation (33) does not admit any positive root of n and so the system is stable for disturbances of all wave numbers. This is in contrast to the Newtonian fluids where the system is stable for stable stratification (CHANDRASEKHAR [1]). If $\beta > 0$ (unstable stratification), the system is stable or unstable if

$$k_x^2 V_A^2 > \frac{g\beta k^2 d^2}{k^2 d^2 + s^2\pi^2} \quad \text{or} \quad k_x^2 V_A^2 < \frac{g\beta k^2 d^2}{k^2 d^2 + s^2\pi^2}. \tag{31}$$

The system is clearly unstable in the absence of a magnetic field. However, the system can be stabilized by large enough magnetic field as can be seen from equation (31), if

$$k_x^2 V_A^2 > \frac{g\beta k^2 d^2}{k^2 d^2 + s^2\pi^2}.$$

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