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A MATHEMATICAL THEOREM IN DOUBLE DIFFUSIVE CONVECTION PROBLEM OF VISCOELASTIC FLUID

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Abstract: The present paper establishes mathematically that viscoelastic thermosolutal convection of Veronis type cannot manifest itself as oscillatory motion of growing amplitude in an initially bottom heavy configuration if the thermal Rayleigh number R_S, the Lewis number τ, the Prandtl number σ and the viscoelastic parameter F satisfy the inequality $R_S \le \frac{27}{4}\pi^4\left\{1+\frac{\tau}{\sigma}(1-F)\right\}, F<1.$ A similar mathematical theorem is also proved for Stern's type configuration.

1. INTRODUCTION

Double diffusive convection, with its archetypal case of heat and salt, generally referred to as thermohaline convection, has been intensively studied in recent past on account of its interesting complexities as well as its direct relevance in many problems of practical interest in the fields of limnology, oceanography, geophysics, astrophysics, chemical engineering, etc. Two fundamental configurations have been studied in this context, the first one by STERN [1] wherein the temperature gradient was stabilizing while the concentration gradient was destabilizing and the second one by VERONIS [2] wherein the temperature gradient was destabilizing while the concentration gradient was stabilizing. The main results derived by Stern and Veronis for their respective problems are that instability might occur in the configurations through a stationary pattern of motions or oscillatory motions provided the destabilizing concentration gradient or temperature gradient is sufficiently large even when the total density field is gravitationally stable. Thus, oscillatory motions of growing amplitude can occur in a thermohaline configuration of Veronis type wherein the total density field is either gravitationally stable or unstable as indicated by the analysis of Veronis notwithstanding the respective character of his work with respect to the nature of the bounding surfaces.

In all the above studies, the fluid has been considered to be Newtonian. However, with the growing importance of non-Newtonian fluids in modern technology and industries, the investigations on such fluids are desirable. The Rivlin–Ericksen [3] fluid is such fluid. Many research workers have paid their attention towards the study of Rivlin–Ericksen fluid. JOHRI [4] has discussed the viscoelastic Rivlin–Ericksen incompressible fluid under time dependent pressure gradient. SISODIA and GUPTA [5] and SRIVASTAVA and SINGH [6] have studied the unsteady flow of a dusty elastico-viscous Rivlin–Ericksen fluid through channel of different cross-sections in the presence of the time dependent pressure gradient. SHARMA and KUMAR [7] have studied the thermal instability of a layer of Rivlin–Ericksen elastico-viscous fluid acted on by a uniform rotation and found that rotation has a stabilizing effect and introduces oscillatory modes in the system. SHARMA and KUMAR [8] have studied the thermal instability in Rivlin–Ericksen elastico-viscous fluid in hydromagnetics.

In the present paper, therefore, an attempt is made to establish a mathematical theorem disproving the existence of neutral or unstable oscillatory motions in an initially bottom heavy/top heavy thermosolutal convection configuration of Veronis/Stern type in a layer of Rivlin–Ericksen viscoelastic fluid.

2. MATHEMATICAL FORMULATION AND ANALYSIS

The relevant governing equations and boundary conditions of thermosolutal convection of a Rivlin–Ericksen viscoelastic fluid are given by

$$(D^2 - a^2)\left[D^2 - a^2 - \frac{p}{\sigma}(1-F)\right]w = Ra^2\theta - R_S a^2 \phi\,, \tag{1}$$

$$(D^2 - a^2 - p)\theta = -w\,, \tag{2}$$

$$\left[(D^2 - a^2) - \frac{p}{\tau}\right]\phi = -\frac{w}{\tau}\,, \tag{3}$$

with

$$w = 0 = \theta = \phi = D^2 w \quad \text{at } z = 0 \text{ and } z = 1 \tag{4}$$

(both boundaries dynamically free),

or

$$w = 0 = \theta = \phi = Dw \quad \text{at } z = 0 \text{ and } z = 1 \tag{5}$$

(both boundaries rigid),

or

$$\left.\begin{aligned} &w = 0 = \theta = \phi = Dw && \text{at } z = 0 \\ \text{and} \quad &w = 0 = \theta = \phi = D^2 w && \text{at } z = 1 \end{aligned}\right\} \tag{6}$$

(lower boundary rigid and upper boundary dynamically free), or

and
$$\left.\begin{array}{l} w = 0 = \theta = \phi = D^2 w \quad \text{at } z = 0 \\ w = 0 = \theta = \phi = Dw \quad \text{at } z = 1 \end{array}\right\} \tag{7}$$

(lower boundary dynamically free and upper boundary rigid),

where z is the real independent variable such that $0 \le z \le 1$, $D = d/dz$ is the differentiation with respect to z, $a^2 > 0$ is a constant, $\sigma > 0$ is a constant, $F > 0$ is a constant, $\tau > 0$ is constant, R and R_S are positive constants for the Veronis configuration and negative constants for Stern's configuration, $p = p_r + ip_i$ is a complex constant in general, such that p_r and p_i are real constants, and as a consequence the dependent variables $w(z) = w_r(z) + iw_i(z)$, $\theta(z) = \theta_r(z) + i\theta_i(z)$, and $\phi(z) = \phi_r(z) + i\phi_i(z)$ are complex valued functions of real variable z such that $w_r(z)$, $w_i(z)$, $\theta_r(z)$, $\theta_i(z)$, $\phi_r(z)$ and $\phi_i(z)$ are real valued functions of real variable z.

The meanings of the symbols from the physical point of view are as follows: z is the vertical co-ordinate, d/dz is the differentiation along the vertical direction, a^2 is the square of the wave number, σ is the Prandtl number, R is the Rayleigh number, R_S is the concentration Rayleigh number, p is the complex growth rate, w is the vertical velocity, θ is the temperature and ϕ is the concentration.

It may further be noted that equations (1)–(7) describe an eigenvalue problem for p and govern thermosolutal instability of Rivlin–Ericksen viscoelastic fluid for any combination of dynamically free and rigid boundaries.

We now prove the following theorem:

Theorem 1: *If* $R > 0$, $R_s > 0$, $p_r \ge 0$, $p_i \ne 0$, $F < 1$ *and* $R_S \le \dfrac{27\pi^4}{4}\left[1 + \dfrac{\tau}{\sigma}(1-F)\right]$,

then a necessary condition for the existence of a non-trivial solution (w, θ, ϕ, p) *of equations* (1)–(3) *together with boundary conditions* (4)–(7) *is*

$$R_S < R. \tag{8}$$

Proof: Multiplying equation (1) by w^* (the complex conjugate of w) throughout and integrating the resulting equation over the vertical range of z, we get

$$\int_0^1 w^*(D^2 - a^2)\left[D^2 - a^2 - \frac{p}{\sigma}(1-F)\right] w\, dz = Ra^2 \int_0^1 w^* \theta\, dz - R_S a^2 \int_0^1 w^* \phi\, dz. \tag{9}$$

Making use of equations (2) and (3), we can write

$$Ra^2 \int_0^1 w^* \theta\, dz = -Ra^2 \int_0^1 \theta (D^2 - a^2 - p^*) \theta^* dz, \tag{10}$$

$$-R_S a^2 \int_0^1 w^* \phi \, dz = R_S a^2 \tau \int_0^1 \phi \left(D^2 - a^2 - \frac{p^*}{\tau} \right) \phi^* dz \,. \tag{11}$$

Combining equations (9)–(11), we obtain

$$\int_0^1 w^* (D^2 - a^2) \left[D^2 - a^2 - \frac{p}{\sigma}(1-F) \right] w \, dz$$

$$= -Ra^2 \int_0^1 \theta (D^2 - a^2 - p^*) \theta^* dz + R_S a^2 \tau \int_0^1 \phi \left(D^2 - a^2 - \frac{p^*}{\tau} \right) \phi^* dz. \tag{12}$$

Integrating the various terms of equation (12) by parts for an appropriate number of times and making use of either of the boundary conditions (4)–(7), it follows that

$$\int_0^1 (|D^2 w|^2 + 2a^2 |Dw|^2 + a^4 |w|^2) \, dz + \frac{p(1-F)}{\sigma} \int_0^1 (|Dw|^2 + a^2 |w|^2) dz$$

$$= Ra^2 \int_0^1 (|D\theta|^2 + a^2 |\theta|^2 + p^* |\theta|^2) \, dz$$

$$- R_S a^2 \tau \int_0^1 \left(|D\phi|^2 + a^2 |\phi|^2 + \frac{p^*}{\tau} |\phi|^2 \right) dz \,. \tag{13}$$

Equating the real and imaginary parts of both sides of equation (13) and cancelling $p_i \neq 0$ throughout from the imaginary part, we get

$$\int_0^1 \left(|D^2 w|^2 + 2a^2 \int_0^1 |Dw|^2 + a^4 |w|^2 \right) dz + \frac{p_r(1-F)}{\sigma} \int_0^1 (|Dw|^2 + a^2 |w|^2) dz$$

$$= Ra^2 \int_0^1 (|D\theta|^2 + a^2 |\theta|^2 + p_r |\theta|^2) dz - R_S a^2 \tau \int_0^1 \left(|D\phi|^2 + a^2 |\phi|^2 + \frac{p_r}{\tau} |\phi|^2 \right) dz, \tag{14}$$

and

$$\frac{(1-F)}{\sigma} \int_0^1 (|Dw|^2 + a^2 |w|^2) dz = -Ra^2 \int_0^1 |\theta|^2 dz + R_S a^2 \int_0^1 |\phi|^2 dz \,. \tag{15}$$

Equation (14) can be written in the alternative form as

$$\int_0^1 (|D^2w|^2 + 2a^2|Dw|^2 + a^4|w|^2)dz + \frac{p_r(1-F)}{\sigma}\int_0^1 (|Dw|^2 + a^2|w|^2)dz$$

$$= Ra^2\int_0^1 (|D\theta|^2 + a^2|\theta|^2)dz - R_S a^2\tau\int_0^1 (|D\phi|^2 + a^2|\phi|^2)dz$$

$$+ p_r a^2\left[R\int_0^1 |\theta|^2 dz - R_S\int_0^1 |\phi|^2 dz\right], \tag{16}$$

and we derive the validity of the theorem from the resulting inequality obtained by replacing each one of the terms of this equation by its appropriate estimate.

We first note that since w, θ and ϕ satisfy $w(0) = 0 = w(1)$, $\theta(0) = \theta(1) = 0$ and $\phi(0) = 0 = \phi(1)$, we have by the Rayleigh–Ritz inequality [9]

$$\int_0^1 |Dw|^2 dz \geq \pi^2 \int_0^1 |w|^2 dz, \tag{17}$$

$$\int_0^1 |D\theta|^2 dz \geq \pi^2 \int_0^1 |\theta|^2 dz, \tag{18}$$

$$\int_0^1 |D\phi|^2 dz \geq \pi^2 \int_0^1 |\phi|^2 dz, \tag{19}$$

and

$$\int_0^1 |D^2w|^2 dz \geq \pi^4 \int_0^1 |w|^2 dz. \tag{20}$$

Utilizing inequalities (17) and (20), we get

$$\int_0^1 (|D^2w|^2 + 2a^2|Dw|^2 + a^4|w|^2)dz \geq (\pi^2 + a^2)^2\int_0^1 |w|^2 dz, \tag{21}$$

Further, since $p_r \geq 0$, therefore, we have

$$\frac{p_r(1-F)}{\sigma}\int_0^1 (|Dw|^2 + a^2|w|^2)dz \geq 0\,. \tag{22}$$

Now, multiplying equation (2) by θ^* (the complex conjugate of θ) throughout and integrating the various terms on the left hand side of the resulting equation by parts for an appropriate number of times by making use of the boundary conditions on θ, namely $\theta(0) = 0 = \theta(1)$, we have from the real part of the final equation

$$\int_0^1 (|D\theta|^2 + a^2|\theta|^2)dz + p_r \int_0^1 |\theta|^2 dz = \text{Real part of} \left(\int_0^1 \theta^* w dz \right)$$

$$\leq \left| \int_0^1 \theta^* w dz \right|$$

$$\leq \int_0^1 |\theta^* w| dz$$

$$\leq \int_0^1 |\theta||w| dz$$

$$\leq \left[\int_0^1 |\theta|^2 dz \right]^{1/2} \cdot \left[\int_0^1 |w|^2 dz \right]^{1/2}$$

(using Schwartz inequality).

Combining this inequality with inequality (18) and the fact that $p_r \geq 0$, we have

$$(\pi^2 + a^2) \int_0^1 |\theta|^2 dz \leq \left[\int_0^1 |\theta|^2 dz \right]^{1/2} \left[\int_0^1 |w|^2 dz \right]^{1/2},$$

which implies that

$$\left[\int_0^1 |\theta|^2 dz \right]^{1/2} \leq \frac{1}{(\pi^2 + a^2)} \left[\int_0^1 |w|^2 dz \right]^{1/2},$$

and thus

$$\int_0^1 (|D\theta|^2 + a^2|\theta|^2)dz < \frac{1}{(\pi^2 + a^2)} \int_0^1 |w|^2 dz. \tag{23}$$

Further, using inequality (19), we have

$$\int_0^1 (|D\phi|^2 + a^2|\phi|^2)dz \geq (\pi^2 + a^2) \int_0^1 |\phi|^2 dz. \tag{24}$$

Also, it follows from equation (15) that

$$R_S a^2 \int_0^1 |\phi|^2 dz \geq \frac{(1-F)}{\sigma} \int_0^1 (|Dw|^2 + a^2 |w|^2) dz . \tag{25}$$

Combining inequalities (24) and (25) and using inequality (17), we get

$$\int_0^1 (|D\phi|^2 + a^2 |\phi|^2) dz \geq \frac{(\pi^2 + a^2)(1-F)}{\sigma R_S a^2} \int_0^1 (|Dw|^2 + a^2 |w|^2) dz$$

$$\geq \frac{(\pi^2 + a^2)^2 (1-F)}{\sigma R_S a^2} \int_0^1 |w|^2 dz . \tag{26}$$

Also, from equation (16) and the fact that $p_r \geq 0$, we obtain

$$p_r a^2 \left(R \int_0^1 |\theta|^2 dz - R_S \int_0^1 |\phi|^2 dz \right) \leq 0 . \tag{27}$$

Now, if permissible, let $R_S \geq R$. Then, in that case, we derive from equation (16) and inequalities (21)–(23), (26) and (27) that

$$\left\{ (\pi^2 + a^2)^2 + \frac{(1-F)\tau(\pi^2 + a^2)^2}{\sigma} - \frac{R_S a^2}{(\pi^2 + a^2)} \right\} \int_0^1 |w|^2 dz < 0 , \tag{28}$$

or

$$\left\{ (\pi^2 + a^2)^2 \left(1 + \frac{(1-F)\tau}{\sigma} \right) - \frac{R_S a^2}{(\pi^2 + a^2)} \right\} \int_0^1 |w|^2 dz < 0 ,$$

or

$$\left\{ \frac{(\pi^2 + a^2)^3}{a^2} \left(1 + \frac{(1-F)\tau}{\sigma} \right) - R_S \right\} \int_0^1 |w|^2 dz < 0 ,$$

which implies that

$$R_S > \frac{(\pi^2 + a^2)^3}{a^2} \left(1 + \frac{(1-F)\tau}{\sigma} \right) ,$$

and thus we necessarily have

$$R_S > \frac{27\pi^4}{4} \left(1 + \frac{(1-F)\tau}{\sigma} \right),$$

since the minimum value of $\frac{(\pi^2 + a^2)^3}{a^2}$ for $a^2 > 0$ is $\frac{27\pi^4}{4}$.

Hence, if $R_S \le \frac{27\pi^4}{4}\left(1+\frac{(1-F)\tau}{\sigma}\right)$, then we must have $R_S < R$, and this completes the proof of the theorem.

Theorem 1 implies from the physical point of view that the thermosolutal convection of Veronis type in the Rivlin–Ericksen viscoelastic fluid cannot manifest itself as an oscillatory motion of growing amplitude in an initially bottom heavy configuration if

$$R_S \le \frac{27\pi^4}{4}\left(1+\frac{(1-F)\tau}{\sigma}\right).$$

Further this result is uniformly valid for the quite general nature of the bounding surfaces.

Special Case 1: For the case when $F = 0$ (Newtonian Fluid) Theorem 1 can be restated as:

If $R > 0$, $R_S > 0$, $p_r \ge 0$, $p_i \ne 0$ and $R_S \le \frac{27\pi^4}{4}\left[1+\frac{\tau}{\sigma}\right]$, then a necessary condition for the existence of a non-trivial solution (w, θ, ϕ, p) of equations (1)–(3) together with boundary conditions (4)–(7) is

$$R_S < R\,.$$

Theorem 2: *If* $R < 0$, $R_S < 0$, $p_r \ge 0$, $p_i \ne 0$, $F < 1$ *and* $|R| \le \frac{27\pi^4}{4}\left[1+\frac{1}{\sigma}(1-F)\right]$, *then a necessary condition for the existence of a non-trivial solution* (w, θ, ϕ, p) *of equations* (1)–(3) *together with boundary conditions* (4)–(7) *is that*

$$|R| < |R_S|\,. \tag{29}$$

Proof: Putting $R = -|R|$, $R_S = -|R_S|$ in equation (1), we have

$$(D^2-a^2)\left[D^2-a^2-\frac{p}{\sigma}(1-F)\right]w = -|R|a^2\theta + |R_S|a^2\phi\,. \tag{30}$$

Multiplying equation (30) by w^* (the complex conjugate of w) throughout and integrating the resulting equation over the vertical range of z, we get

$$\int_0^1 w^*(D^2-a^2)\left[D^2-a^2-\frac{p}{\sigma}(1-F)\right]w\,dz = -|R|a^2\int_0^1 w^*\theta dz + |R_S|a^2\int_0^1 w^*\phi dz\,. \tag{31}$$

Making use of equations (2) and (3), we can write

$$-|R|a^2\int_0^1 w^*\theta dz = |R|a^2\int_0^1 \theta(D^2-a^2-p^*)\theta^* dz\,, \tag{32}$$

$$|R_S|a^2\int_0^1 w^*\phi dz = -|R_S|a^2\tau\int_0^1 \phi\left(D^2-a^2-\frac{p^*}{\tau}\right)\phi^* dz. \tag{33}$$

Combining equations (31)–(33), we obtain

$$\int_0^1 w^*(D^2-a^2)\left[D^2-a^2-\frac{p}{\sigma}(1-F)\right]w\,dz = |R|a^2\int_0^1 \theta(D^2-a^2-p^*)\theta^* dz$$

$$-|R_S|a^2\tau\int_0^1 \phi\left(D^2-a^2-\frac{p^*}{\tau}\right)\phi^* dz. \tag{34}$$

Integrating the various terms of equation (34) by parts for an appropriate number of times and making use of either of the boundary conditions (4)–(7), we get

$$\int_0^1 (|D^2w|^2+2a^2|Dw|^2+a^4|w|^2)\,dz+\frac{p(1-F)}{\sigma}$$

$$\int_0^1 (|Dw|^2+a^2|w|^2)\,dz = -|R|a^2\int_0^1 (|D\theta|^2+a^2|\theta|^2+p^*|\theta|^2)\,dz$$

$$+|R_S|a^2\tau\int_0^1 \left(|D\phi|^2+a^2|\phi|^2+\frac{p^*}{\tau}|\phi|^2\right)dz. \tag{35}$$

Equating the real and imaginary parts of both sides of equation (35) and cancelling $p_i \neq 0$, throughout from the imaginary part, we get

$$\int_0^1 (|D^2w|^2+2a^2|Dw|^2+a^4|w|^2)\,dz+\frac{p_r(1-F)}{\sigma}$$

$$\int_0^1 (|Dw|^2+a^2|w|^2)\,dz = -|R|a^2\int_0^1 (|D\theta|^2+a^2|\theta|^2+p_r|\theta|^2)\,dz$$

$$+|R_S|a^2\tau\int_0^1 \left(|D\phi|^2+a^2|\phi|^2+\frac{p_r}{\tau}|\phi|^2\right)dz, \tag{36}$$

and

$$\frac{1-F}{\sigma}\int_0^1 (|Dw|^2+a^2|w|^2)\,dz = |R|a^2\int_0^1 |\theta|^2 dz - |R_S|a^2\int_0^1 |\phi|^2 dz. \tag{37}$$

Equation (36) can be written in the alternative form as

$$\int_0^1 (|D^2w|^2 + 2a^2|Dw|^2 + a^4|w|^2)\,dz + \frac{p_r(1-F)}{\sigma}\int_0^1 (|Dw|^2 + a^2|w|^2)\,dz$$

$$= -|R|a^2\int_0^1 (|D\theta|^2 + a^2|\theta|^2)\,dz + |R_S|a^2\tau\int_0^1 (|D\phi|^2 + a^2|\phi|^2)\,dz$$

$$+ p_r a^2\left[|R_S|\int_0^1 |\phi|^2 dz - |R|\int_0^1 |\theta|^2 dz\right], \tag{38}$$

and we derive the validity of the theorem from the resulting inequality obtained by replacing each one of the terms of this equation by its appropriate estimate.

Utilizing inequalities (17) and (20), we get

$$\int_0^1 (|D^2w|^2 + 2a^2|Dw|^2 + a^4|w|^2)\,dz \ge (\pi^2 + a^2)^2\int_0^1 |w|^2 dz\,. \tag{39}$$

Further, since $p_r \ge 0$, therefore, we have

$$\frac{p_r(1-F)}{\sigma}\int_0^1 (|Dw|^2 + a^2|w|^2)\,dz \ge 0\,. \tag{40}$$

Now, multiplying equation (3) by ϕ^* (the complex conjugate of ϕ) and integrating the various terms on the left hand side of the resulting equation by parts for an appropriate number of times by making use of boundary conditions on ϕ, namely $\phi(0) = 0 = \phi(1)$, we have from the real part of the final equation

$$\int_0^1 (|D\phi|^2 + a^2|\phi|^2)\,dz + \frac{p_r}{\tau}\int_0^1 |\phi|^2 dz = \text{Real part of}\left(\frac{1}{\tau}\int_0^1 \phi^* w\,dz\right)$$

$$\le \left|\frac{1}{\tau}\int_0^1 \phi^* w\,dz\right|$$

$$\le \frac{1}{\tau}\int_0^1 |\phi^* w|\,dz$$

$$\le \frac{1}{\tau}\int_0^1 |\phi||w|\,dz$$

$$\le \frac{1}{\tau}\left[\int_0^1 |\phi|^2 dz\right]^{1/2}\cdot\left[\int_0^1 |w|^2 dz\right]^{1/2}$$

(using Schwartz inequality).

Using inequality (19) and the fact that $p_r \geq 0$, in the above inequality, we have

$$(\pi^2 + a^2)\int_0^1 |\phi|^2 dz \leq \frac{1}{\tau}\left[\int_0^1 |\phi|^2 |w| dz\right]^{1/2}\left[\int_0^1 |w|^2 dz\right]^{1/2},$$

which implies that

$$\left[\int_0^1 |\phi|^2 dz\right]^{1/2} \leq \frac{1}{\tau(\pi^2 + a^2)}\left[\int_0^1 |w|^2 dz\right]^{1/2}$$

and thus

$$\int_0^1 (|D\phi|^2 + a^2 |\phi| dz \leq \frac{1}{\tau(\pi^2 + a^2)}\int_0^1 |w|^2 dz. \tag{41}$$

Further, using inequality (18), we have

$$\int_0^1 (|D\theta|^2 + a^2 |\theta|^2 dz \geq (\pi^2 + a^2)\int_0^1 |\theta|^2 dz. \tag{42}$$

Also, it follows from equation (15) that

$$|R| a^2 \int_0^1 |\theta|^2 dz \geq \frac{1-F}{\sigma}\int_0^1 (|Dw|^2 + a^2 |w|^2) dz. \tag{43}$$

Combining inequalities (42) and (43), and using inequality (17), we get

$$\begin{aligned}\int_0^1 (|D\theta|^2 + a^2 |\theta|^2) dz &\geq \frac{(\pi^2 + a^2)(1-F)}{|R| a^2 \sigma}\int_0^1 (|Dw|^2 + a^2 |w|^2) dz \\ &\geq \frac{(\pi^2 + a^2)^2 (1-F)}{|R| a^2 \sigma}\int_0^1 |w|^2 dz.\end{aligned} \tag{44}$$

Also, from equation (38) and the fact that $p_r \geq 0$, we obtain

$$p_r a^2\left(-|R|\int_0^1 |\theta|^2 dz + |R_S|\int_0^1 |\phi|^2 dz\right) \leq 0. \tag{45}$$

Now, if permissible, let $|R| \geq |R_S|$. Then in that case, we derive from equation (38) and inequalities (39)–(41), (44) and (45) that

$$\left[(\pi^2+a^2)^2+\frac{(1-F)(\pi^2+a^2)^2}{\sigma}-\frac{|R|a^2}{(\pi^2+a^2)}\right]\int_0^1|w|^2dz<0, \tag{46}$$

or

$$\left[(\pi^2+a^2)^2\left\{1+\frac{(1-F)}{\sigma}\right\}-\frac{|R|a^2}{(\pi^2+a^2)}\right]\int_0^1|w|^2dz<0,$$

or

$$\left[\frac{(\pi^2+a^2)^3}{a^2}\left\{1+\frac{(1-F)}{\sigma}\right\}-|R|\right]\int_0^1|w|^2dz<0,$$

which implies that

$$|R|>\frac{(\pi^2+a^2)^3}{a^2}\left(1+\frac{(1-F)}{\sigma}\right),$$

and thus we necessarily have

$$|R|>\frac{27\pi^4}{4}\left(1+\frac{(1-F)}{\sigma}\right),$$

since the minimum value of $\frac{(\pi^2+a^2)^3}{a^2}$ for $a^2>0$ is $\frac{27\pi^4}{4}$.

Hence, if $|R|\le\frac{27\pi^4}{4}\left(1+\frac{(1-F)}{\sigma}\right)$, then we must have

$$|R|<|R_S|,$$

and this completes the proof of the theorem.

Theorem 2 implies from the physical point of view that the thermosolutal convection of Sterns type in the Rivlin–Eriksen viscoelastic fluid cannot manifest itself as an oscillatory motion of growing amplitude in an initially top heavy configuration if

$$|R|\le\frac{27\pi^4}{4}\left(1+\frac{(1-F)}{\sigma}\right).$$

Further this result is uniformly valid for the quite general nature of the bounding surfaces.

Special Case 2: For the case when $F = 0$ (Newtonian Fluid) Theorem 2 can be restated as:

If $R < 0$, $R_S < 0$, $p_r \geq 0$, $p_i \neq 0$ and $|R| \leq \frac{27\pi^4}{4}\left(1+\frac{1}{\sigma}\right)$, then a necessary condition for the existence of a non-trivial solution (w, θ, ϕ, p) of equations (1)–(3) together with boundary conditions (4)–(7) is that

$$|R| < |R_S|.$$

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