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RELIABILITY ASSESSMENT OF BEARING CAPACITY OF LAYERED SOILS USING HIGH DIMENSIONAL MODEL REPRESENTATION (HDMR)

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Abstract: HDMR (High Dimensional Model Representation) is a relatively new method that is used to form response surface based on results obtained through laboratory experiments or through numerical calculations. So far the method has been used mainly in chemistry, although a few studies conducted in recent years show that it can be considered a useful tool in soil mechanics and foundation engineering.

The subject matter of this paper is the application of HDMR method to reliability assessment of bearing capacity of layered soils. Madej's method, widely recognized and used by Polish engineers, is applied to conduct the calculations. In the analysed case bearing capacity is not expressed by means of an explicit formula.

To fit the approximate functions of bearing capacity, its values are calculated on a grid of points equally spread on ranges of variables. Finding the relation between input and output data is conducted by means of assessing each variable's influence on response's mean value within a given scope.

Approximate functions have been used to calculate reliability indices by means of FORM, SORM and Monte Carlo methods.

1. INTRODUCTION

Although there are many numerical procedures and tools dedicated to solving reliability problems in geotechnics, there is still a need for more efficient and accurate methods. This need for new algorithms is caused by a relatively high uncertainty and heterogeneity of soil parameters when compared to other construction engineering problems. Finite Element Method (FEM) and Finite Difference Method (FDM) are considered effective tools yielding good results in simple cases. However, in order to be useful in reliability assessment, they require use of approximation methods, such as Response Surface Method (RSM) ([1], [17]). Although RSM can be very efficient when solving simple problems, the computational effort increases quickly when the number of variables is four or greater.

Recently a new method called High Dimensional Model Representation (HDMR) has been proposed to improve efficiency of reliability computation. The method has been developed by researches in applied mathematics and chemistry. The most important papers were given by Rabitz and Omer [18], Sobol [23], Demiralp [4], Kaya et al. [10], Shorter, Ip and Rabitz [22], Li et al. [11]. Lately there appeared some applications to civil engineering problems (e.g., [15], [16], [20], [21]). Chowdhury and Rao [2], [3] were the first authors to apply HDMR to geotechnical problems, however their suggestions were limited to slope stability problems. By now no applications to probabilistic assessments in other important branches of geotechnical engineering have been reported.

Probabilistic approaches to bearing capacity of shallow foundation have been exhaustively elaborated in the literature ([6], [24]). There is, however, no probabilistic assessment dealing with layered soils. The reason for this is the lack of simple analytical deterministic solutions for bearing capacity of layered soils.

In the present paper, the authors propose a probabilistic bearing capacity evaluation of shallow foundation resting of layered soil by means of HDMR method. A numerical algorithm has been developed in order to assess bearing capacity using analytical solution published by Madej [13], [14] – a method used for designing foundations in Poland for decades.

2. BEARING CAPACITY EVALUATION FOR LAYERED SOILS – MADEJ'S METHOD

An algorithm for estimating bearing capacity of shallow foundations was proposed by Madej [13], [14]. Its use is limited to cases in which two layers (A) of soil with good strength parameters are separated by a relatively thin layer (B) of weaker soil (weak soil's $h < 2B$) as presented in Fig. 1. This method has been used by engineers in Poland for decades and has proved itself trustworthy.

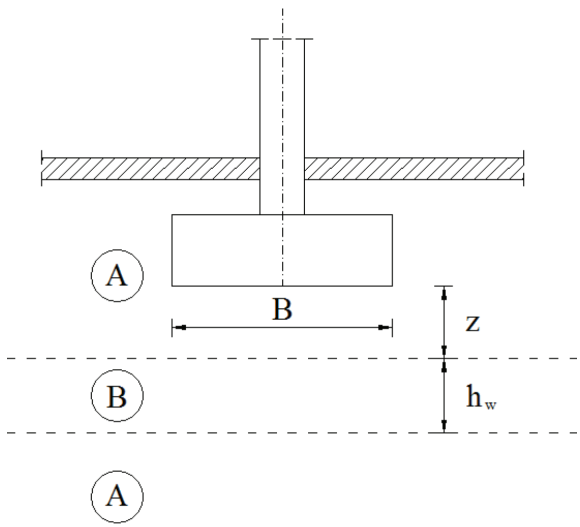

Fig. 1. Schematic presentation of Madej's method assumptions

In this case, bearing capacity of the foundation is given by the following equation

$$Q_T = Q_W + \eta_m (Q_S - Q_W) \quad \text{when} \quad h_W \le 0.5B, \tag{1}$$

$$Q_W = Q_{1W} + \eta_m (Q_S - Q_{1W}) \quad \text{when} \quad h_W > 0.5B, \tag{2}$$

$$Q_{1W} = Q_W + (Q_S - Q_W)\left(1 - \frac{2h_W}{B}\right)^2, \tag{3}$$

where

h_W – weak soil thickness,

B – foundation width,

Q_T – total bearing capacity of layered soil,

Q_W – bearing capacity of homogeneous soil with weak layer parameters,

Q_S – bearing capacity of homogeneous soil with strong layer parameters,

η_m – Madej's coefficient.

Coefficients η_m have no analytical representation and have to be found by means of nomograms, as shown in Figs. 2–4.

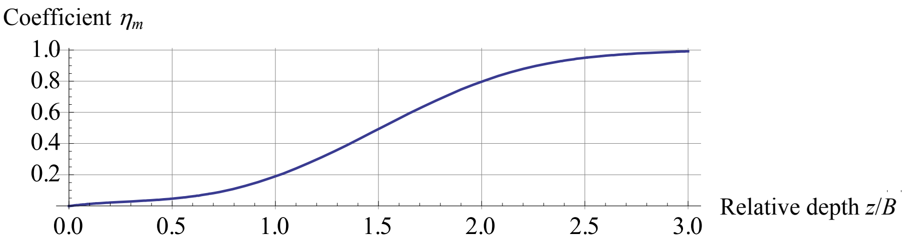

Fig. 2. Coefficients η_m versus relative depth z/B in the cases where $\frac{h_W}{B} \le 0.5$

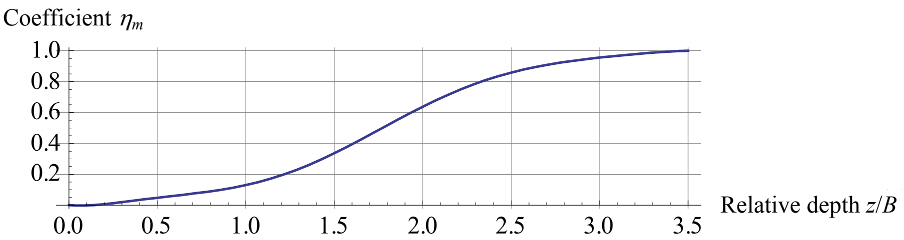

Fig. 3. Coefficients η_m versus relative depth z/B in the cases where $0.5 < \frac{h_W}{B} < 1.0$

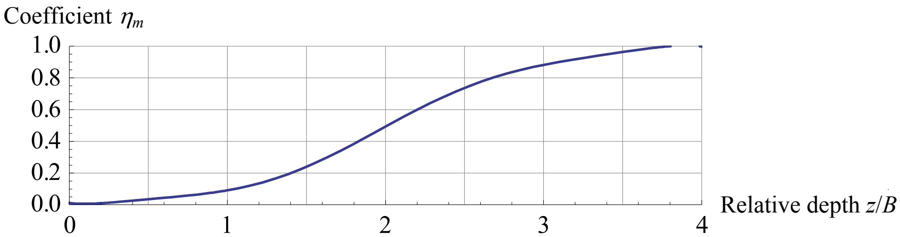

Fig. 4. Coefficients η_m versus relative depth z/B in the cases where $\frac{h_W}{B} \geq 1.0$

In order to facilitate the use of nomograms, they have been approximated by appropriately chosen functions using the least squares method.

3. HIGH DIMENSIONAL MODEL REPRESENTATION

The general idea of High Dimensional Model Representation is to capture high-dimensional relationships between sets of input and output model variables. It is an efficient tool that provides descriptions of multivariate models. Starting from a constant and adding higher, consecutive order terms, the HDMR response gradually approaches the original state function. In this paper, the number of random variables is set to $N = 3$ for approximation of homogenous soil and $N = 6$ for non-homogenous soil (Madej's method). A short description of the method is given below.

Let $x = \{x_1, x_2, ..., x_N\}^T$ represent the vector of N variables and the model response variable be $g(x)$. The response function can be described as

$$g(x) = g_0 + \sum_{i=1}^{N} g(x_i) + \sum_{1 \leq i_1 \leq i_2 \leq N}^{N} g_{i_1 i_2}(x_{i_1}, x_{i_2})$$

$$+ \ldots + \sum_{1 \leq i_1 < \ldots < i_k \leq N}^{N} g_{i_1 i_2 - i_k}(x_{i_1}, x_{i_2}, ..., x_{i_k}) + g_{12\_N}(x_1, x_2, ..., x_N) \qquad (4)$$

where

- g_0 – value of function g calculated at the central point c and it is referred to as the mean response or zeroth-order (the central point is usually selected as the point of expected values of random variables if probabilistic evaluations are carried out),
- $g_i(x_i)$ – first-order function, where x_i is the only one variable to change, while the rest of coordinates remain constant at their mean values (central point values). The sum of g_0 and all the functions g_i is called the first order expansion,

- $g_{i_1 i_2}(x_1, x_2)$ – second-order function that describes cooperative effect of x_{i_1} and x_{i_2} on output $g(x)$. The rest of coordinates remain constant and equal to their central point values,
- $g_{i_1 i_2 \_ i_k}(x_{i_1}, x_{i_2}, ..., x_{i_k})$ – higher-order terms, describing the cooperative effect of x_{i_1}, x_{i_2}, ..., x_{i_k} on the output $g(x)$,
- $g_{12\_N}(x_1, x_2, ..., x_N)$ – last element of expansion, representing any residual dependence of all input variables on the output $g(x)$.

In this particular study, ranges of variables were chosen to correspond to their probabilistic distributions. The reference point $c = \{c_1, c_2, ..., c_N\}^T$ was selected and its coordinates were set to variables' mean values (expected values). The approximation functions are evaluated with respect to the defined point c, and the relationships between components in the expansion are described as

$$g_0 = g(c), \tag{5}$$

$$g_i(x_i) = g(x_i, c^i) - g_0, \tag{6}$$

$$g_{i_1 i_2}(x_{i_1}, x_{i_2}) = g(x_{i_1}, x_{i_2}, c^{i_1, i_2}) - g_{i_1}(x_{i_1}) - g_{i_2}(x_{i_2}) - g_0, \tag{7}$$

where

$$g(x_i, c^i) = g(c_1, c_2, ..., c_{i-1}, x_i, c_{i+1}, ..., c_N).$$

Removing higher-order terms from equation (4) results in obtaining first-order and second-order expansions

$$g(x) = g_0 + \sum_{i=1}^{N} (g(x_i, c^i) - g(c)) + \Re_2, \tag{8}$$

$$g(x) = g_0 + \sum_{i=1}^{N} (g(x_i, c^i) - g(c))$$
$$\sum_{1 \le i_1 \le i_2 \le N} [(g_{i_1 i_2}(x_{i_1}, x_{i_2}, c^{i_1, i_2}]) - g(x_{i_1}) - g(x_{i_2}) - g(c)) + \Re_3, \tag{9}$$

with $\Re_2$ and $\Re_3$ being the residual errors.

Application of equations (5)–(7) to equations (8) and (9) leads to

$$g(x) = \sum_{i=1}^{N} g(x_i, c^i) - (N-1) g(c) + \Re_2, \tag{10}$$

$$g(x) = \sum_{1 \le i_1 \le i_2 \le N} (g_{i_1}(x_{i_1}, x_{i_2}, c^i)) - (N-2)\sum_{i=1} g_i(x_i, c^i) + \frac{(N-1)(N-2)}{2} g(c) + \Re_3, \tag{11}$$

Therefore, first and second order approximations of function $g(x)$ are respectively given by

$$\tilde{g}(x) = \sum_{i=1}^{N} g(x_i, c^i) - (N-1)g(c), \tag{12}$$

$$\tilde{g}(x) = \sum_{1 \le i_1 \le i_2 \le N} (g_{i_1}(x_{i_1}, x_{i_2}, c^{i_1,i_2})) - (N-2)\sum_{i=1} g_i(x_i, c^i) + \frac{(N-1)(N-2)}{2} g(c). \tag{13}$$

Application of HDMR begins with calculation of $g(c)$ value. Next, the sample points should be chosen regularly and symmetrically throughout the ranges of variables in order to provide satisfactory precision, while using only first and second order expansions. In this paper, the coordinates of sample points were calculated as

$$x_i^k = x_k^1 + k * \frac{x_i^n - x_i^1}{n}, \qquad k = 0, 1, ..., n. \tag{14}$$

It is assumed that x_i^k values are in ascending order and n is the number of sample points along axis.

The accuracy of the method strictly depends on the form of approximate functions. In this paper, polynomials of the third and fourth degree were used and the results obtained were later compared. Least squares method was applied to fit the polynomials to the original function $g(x)$

$$g(x_i, c^i) \approx \Phi(x_i, a_i), \tag{15}$$

$$\Phi(x_i, a_i) = \sum_{k=1}^{p} \alpha_{i,k} x_i^{k-1}, \tag{16}$$

where α_i are the coefficients and p is the degree of polynomials. In both examples shown in this study, second-order approximate functions were used

$$\tilde{g}(x) = \sum_{1 \le i_1 \le i_2 \le N}^{N} \Phi_{i_1 i_2}(x_{i_1}, x_{i_2}, \alpha_{i_1 i_2}) - (N-2)\sum_{i=1}^{N} \Phi(x_i, \alpha_i) + \frac{(N-1)(N-2)}{2} g_0. \tag{17}$$

4. APPLICATION OF HIGH DIMENSIONAL MODEL REPRESENTATION TO RELIABILITY ANALYSIS OF HOMOGENEOUS SOIL USING POLISH STANDARD PN-83/B-03020

A simple example has been chosen to show the general idea and precision that can be obtained by means of HDMR algorithm. Shallow foundation of infinite length is based on a homogeneous soil consisting of clay. In this case, the limit state function is given in explicit form as follows

$$q = \left[\left(1+0.3\frac{B}{L}\right)*N_c*c+\left(1+1.5\frac{B}{L}\right)*N_D*\gamma*D_{\min}+\left(1-0.25\frac{B}{L}\right)*N_B*\gamma*B\right], \quad (18)$$

B = 1.5 m – foundation width,
L = 15 m – foundation length,
$D_{\min}$ = 1 m,
Coefficients N are given by:

$$N_D = e^{\pi*\mathrm{tg}\phi}\mathrm{tg}^2\left(\frac{\pi}{4}+\frac{\phi}{2}\right),$$

$$N_C = (N_D-1)*\mathrm{ctg}\phi,$$

$$N_B = 0.75*(N_D-1)*\mathrm{tg}\phi.$$

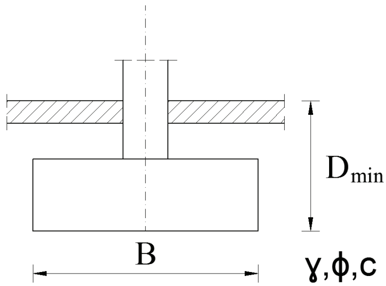

Fig. 5. Schematic presentation of PN-83/B-03020 situation and denotations

The soil parameters (cohesion, friction angle and unit weight) are all considered as beta distributed random variables (Table 1).

The second order HDMR expansions, consisting of third and fourth degree polynomials, were used to create limit state functions (The second order expansion functions applied in the calculations use polynomials of up to sixth and eighth degree, respectively). As a result reliability indexes β, corresponding to certain applied loads, were calculated using FORM, SORM and Monte Carlo (10^6 realisations), as shown in Figs. 6 and 7.

Table 1

Soil parameters

Parameter	Expected value μ	Standard deviation σ	Coefficient of variation ν	x_{min}	x_{max}
Cohesion (c)	30 kPa	6 kPa	0.2	15 kPa	45 kPa
Friction angle (ϕ)	18°	1.8°	0.1	12°	24°
Weight (γ)	$21\frac{\text{kN}}{\text{m}^3}$	$0.84\frac{\text{kN}}{\text{m}^3}$	0.04	$19\frac{\text{kN}}{\text{m}^3}$	$23\frac{\text{kN}}{\text{m}^3}$

The reliability analysis was performed for HDMR expansions (third degree polynomials) and the original function, given explicitly in the Polish Standard. The results are shown in Table 2.

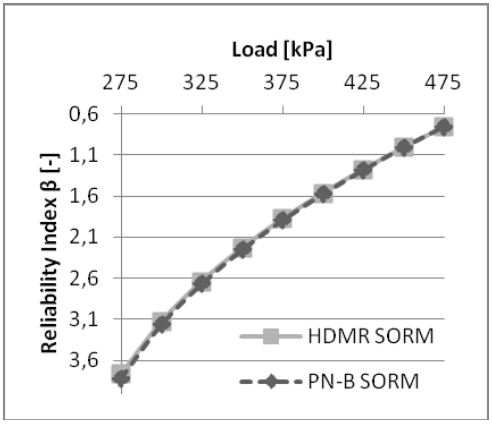

Fig. 6. Reliability index β versus static load. Limit state function approximated by third degree polynomials

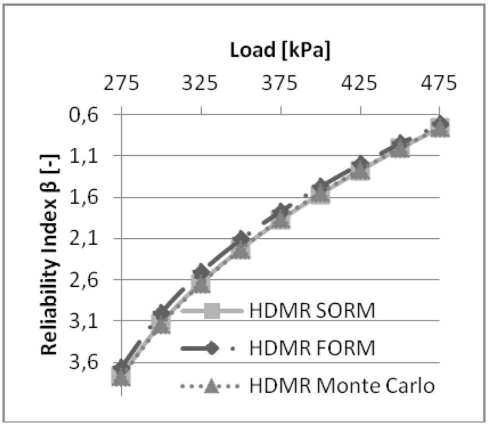

Fig. 7. Reliability index β versus static load. Limit state function approximated by fourth degree polynomials

Table 2

Comparison of reliability indices resulting from HDMR and explicit utilisation of the bearing capacity formula

q [kPa]	275	300	325	350	375	400	425	450	475	500	525
HDMR SORM β	3.77	3.13	2.64	2.23	1.88	1.57	1.28	1.01	0.76	0.53	0.30
HDMR MC β	3.78	3.15	2.68	2.26	1.89	1.57	1.29	1.02	0.77	0.53	0.30
PN-B SORM β	3.81	3.16	2.66	2.25	1.89	1.57	1.28	1.01	0.76	0.52	0.30

The above example proves that the approximation by means of HDMR method yields accurate results. The results obtained in the analysis with polynomials of 3rd and 4th degrees are similar, which shows sufficient accuracy of the solution.

5. APPLICATION OF HIGH DIMENSIONAL MODEL REPRESENTATION TO RELIABILITY ANALYSIS OF NON-HOMOGENEOUS SOIL USING MADEJ'S METHOD COMBINED WITH EUROCODE 7

Results in the previous section have demonstrated efficiency and good accuracy of the approach proposed. The purpose of the present section is to apply HDMR approximation in order to carry out reliability assessment of bearing capacity of layered soil. Moreover, the problem analysed will show HDMR functionality in multivariate cases. A shallow foundation based on a non-homogenous soil is analyzed. Its dimensions and assumptions are as follows: B = 1.5 m, L = 15 m, h_w = 1 m, z = 1 m (see Fig. 1). Total number of variables is N = 6 (cohesion, friction angle and weight for each kind of soil), as listed in Table 3.

Table 3

Probabilistic characteristics of random variables involved in the problem of layered soil

Parameter		μ	σ	v	x_{min}	x_{max}
Strong soil	Cohesion (c_{strong})	30 kPa	6 kPa	0.2	15 kPa	45 kPa
	Friction angle (ϕ_{strong})	18°	1.8°	0.1	12°	24°
	Soil weight (γ_{strong})	$21\ \frac{kN}{m^3}$	$0.84\ \frac{kN}{m^3}$	0.04	$19\ \frac{kN}{m^3}$	$23\ \frac{kN}{m^3}$
Weak soil	Cohesion (c_{weak})	15 kPa	3 kPa	0.2	5 kPa	25 kPa
	Friction angle (ϕ_{weak})	10°	1°	0.1	6°	14°
	Soil weight (γ_{weak})	$20\ \frac{kN}{m^3}$	$0.8\ \frac{kN}{m^3}$	0.04	$18\ \frac{kN}{m^3}$	$22\ \frac{kN}{m^3}$

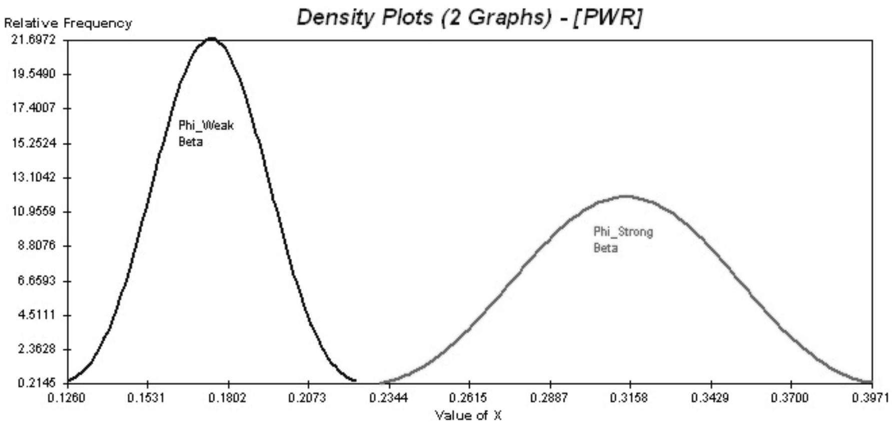

Fig. 8. Probability density of friction angles (φ_{strong}, φ_{weak})

Random variables were treated as mutually independent with beta distributions. As an example probability density functions of strong and weak layers internal friction angles are plotted on the graphs in Fig. 8.

The second order HDMR expansions, consisting of third and fourth degree polynomials, were used to create limit state functions. As a result reliability indexes β, corresponding to certain applied loads, were calculated using FORM, SORM and Monte Carlo (10^6 realisations), as shown in Table 4 and Figs. 9, 10.

Table 4

Results obtained by means of HDMR

Function	Fourth degree polynomials			Third degree polynomials		
Load [kPa]	β FORM	β SORM	β MC	β FORM	β SORM	β MC
160	3.827	4.075	4.049	3.646	3.943	3.945
170	3.361	3.591	3.592	3.136	3.406	3.446
180	2.953	3.165	3.212	2.714	2.965	3.033
190	2.585	2.779	2.781	2.349	2.58	2.588
200	2.244	2.42	2.429	2.022	2.233	2.239
210	1.923	2.08	2.081	1.72	1.914	1.909
220	1.616	1.753	1.741	1.438	1.613	1.598
230	1.318	1.436	1.436	1.169	1.326	1.328
240	1.028	1.125	1.129	0.9086	1.043	1.047
250	0.743	0.820	0.8215	0.6546	0.7718	0.7707
260	0.461	0.518	0.518	0.4041	0.5035	0.4952
270	0.187	0.224	0.2206	0.1554	0.2364	0.2218
280	–0.042	–0.060	–0.061	–0.0931	–0.0198	–0.014
290	–0.352	–0.347	–0.341	–0.3424	–0.2904	–0.3075
300	–0.609	–0.621	–0.6221	–0.5932	–0.5618	–0.5818

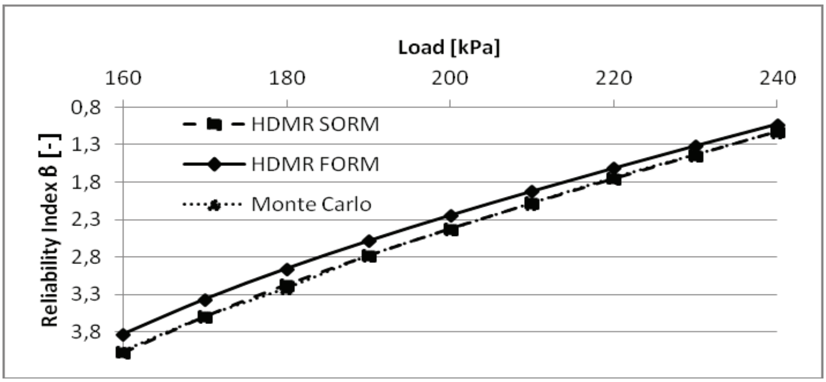

Fig. 9. Reliability index β versus static load. Limit state function approximated by fourth degree polynomials

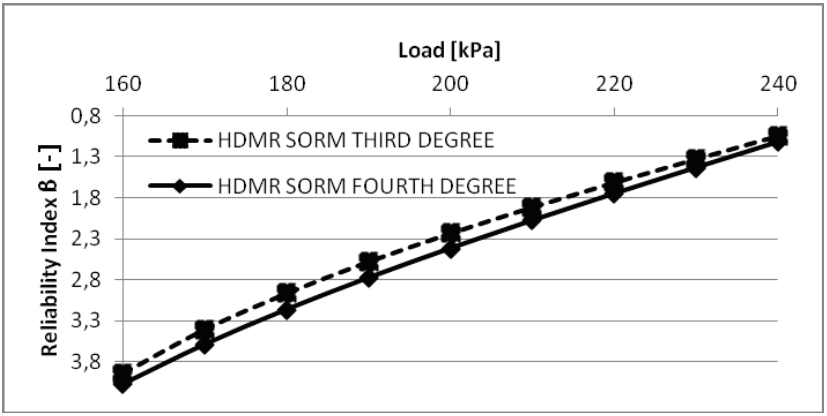

Fig. 10. Reliability index β versus static load. Comparison between results obtained by use of third and fourth degree approximations

6. CONCLUSION

In this paper, the authors have presented HDMR as an efficient method for analyzing reliability indexes of shallow foundations on layered soil. The results have been obtained using Madej's method – a method based on model analysis, used in Poland for decades – together with formulas presented in Eurocode 7.

This way an efficient algorithm for reliability evaluations concerning bearing capacity of layered soils has been created. A comparative example for the case of homogeneous soil, where reliability computations have been carried out based both on analytical solution and HDMR approximation, shows high coincidence of results. Therefore, one can expect that the approximate solution for layered soil based on HDMR yields results of sufficient accuracy.

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